Dynamic financial planning for a household in a multi-period optimization approach

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Abstract

We discuss an optimization model to obtain an optimal investment and insurance strategy for a household. In this paper, we extend the studies in Hibiki, Komoribayashi and Toyoda(2005) for the practical use, and propose the model with three factors we need to consider if the householder is dead. Three factors are receipt of survivor’s pension, exemption from housing loan payments, and change of the consumption level. We examine the additional effect by three factors with numerical examples. We analyze the sensitivity of parameters associated with home buying in order to examine the home buying strategy. Moreover, We aggregate the paths we may not use to calculate the downside risk, and we formulate the simplified model which can be solved faster than the original model.

keywords: multi-period optimization, financial planning for a household, optimal investment strategy, life insurance, home buying strategy

1 Introduction

We discuss an optimization model to obtain an optimal investment and insurance strategy for a household. Recently, financial institutions have promoted giving a financial advice for individual investors. How much will the household need to save when the householder retires? What kind of financial products should be purchased to hedge various risk such as market risk, inflation risk, and catastrophe insurance risk? Financial institutions need to recommend appropriate financial products to answer these questions in conjunction with a life cycle, current asset, and future income.

We clarify how a set of asset mix, life insurance, and fire insurance affect asset and liability management for a household. We develop a multi-period optimization model which involves determining a set of financial products, hedging risk associated with a life cycle of the household and saving for the old age, and the simulated path approach (Hibiki, 2001b) can be used to solve this problem.


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optimization approach. Hibiki, Komoribayashi and Toyoda (2005) describe a multi-period optimization model to determine an optimal set of asset mix, life insurance and fire insurance in conjunction with their life cycle and characteristics. The model is examined with numerical examples. In addition, some financial advices for three households are illustrated for practical use, and the results which coincide with the practical feeling are obtained.

Risk associated with the householder’s death is hedged by life insurance. In this paper, we extend the studies in Hibiki, Komoribayashi and Toyoda (2005) for the practical use, and propose a model with three factors we need to consider if the householder is dead. Three factors are receipt of survivor’s pension, exemption from the home loan payments, and change of the consumption level. We examine additional effects by three factors with numerical examples.

Given four kinds of parameters associated with home buying, or time, down payment, loan period, and mortgage interest rate, we solve the problems. We analyze the sensitivity of these parameters in order to examine the home buying strategy.

It takes much time to solve the large-scale problems with a long planning period. For example, it is assumed that the householder is thirty years old, and will retire thirty years after. When one period is one year and the number of paths is five thousand, the number of the planning period is thirty periods, both numbers of constraints and decision variables are about 150,000, and it takes about six minutes to solve a problem. It is necessary to solve the problems faster in order to examine the model with a lot of numerical examples. If we calculate a downside risk measure, we do not use paths where a terminal wealth is larger than a target associated with the downside risk. We aggregate paths we may not use to calculate the downside risk, and we formulate a simplified model which can be solved faster than the original model.

This paper is organized as follows. We introduce the concept of the simulated path approach in Section 2. We define a household and three kinds of financial products, or securities, life insurance, and fire insurance to describe the model structure. We consider three factors in addition to Hibiki, Komoribayashi, and Toyoda (2005) for practical use: ① survivor’s pension, ② exemption from housing loan payments, ③ change of the consumption level. Section 3 shows the formulation of the multi-period ALM optimization model for a household. We demonstrate some numerical tests in Section 4. Section 5 shows the sensitivity analysis of the parameters associated with home buying in order to examine the home buying strategy. Section 6 shows the formulation using a grouped path and some numerical tests in order to solve problems fast. Section 7 provides our concluding remarks.

2 Solution technique and model structure

At first, we explain a multi-period optimization in the simulated path approach to determine an optimal set of asset mix and insurance. Next, we define a household, and describe the income and the consumption expense. We clarify the characteristics of financial instruments such as securities, life insurance and fire insurance.

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1 Problems are solved using NUOPT (Ver. 7.1.5) – mathematical programming software package developed by Mathematical System, Inc. – on Windows XP personal computer which has 2.13 GHz CPU and 2GB memory. All of the problems in this paper are also solved using the same computer and software.
2.1 Simulated path approach

It is critical for stochastic modeling to handle uncertainties and investment decisions appropriately. The decisions have to be independent from knowledge of actual paths that will occur. Thus, we must define a set of decision variables and a set of constraints to prevent an optimization model from being solved by anticipating events in the future. In addition, we need a sufficient number of paths to get a better accuracy with respect to the future possible events.

The concept of scenarios is typically employed for modeling random parameters in the multi-period stochastic programming models. Scenarios are constructed via a tree structure as in the left-hand side of Figure 1 (see Mulvey and Ziemba, 1995 and 1998 for a detailed discussion). The model is based on the expansion of the decision space, taking into account a conditional nature of the scenario tree. Conditional decisions are made at each node, subject to the modeling constraints. To ensure that the constructed representative set of scenarios covers the set of possibilities to a sufficient degree, the numbers of decision variables and constraints in the scenario tree may grow exponentially. This model is called a scenario tree model.

Meanwhile, simulated paths give another description of scenarios shown as in the right-hand side of Figure 1. Hibiki [2000, 2001b] developed a simulated path model in a multi-period optimization framework. If we formulate stochastic differential equations or time series models associated with asset returns, discrete asset returns are generated by a standard Monte Carlo simulation technique to describe uncertainties more accurately than would the scenario tree as in the left-hand side of Figure 1. However, if a decision is made on the associated path, the model is solved anticipating the event in the future. Therefore, the rule that the same investment decision is made at each time is defined to satisfy the non-anticipativity condition in the simulated path model.

We do not use the scenario tree model, but the simulated path model. This is because the model involves determining a set of optimal life and fire insurance money, and we need a lot of
paths at each time to describe the mortality rate and the rate of the fire. According to the life insurance standard life table (1996), the mortality rate for 50 years old men is 0.379%. Even if the householder dies on a path to describe the appropriate mortality rate, we need 264 paths at each time. If we construct a scenario tree over 30 years, we have to generate an enormous tree, and we cannot solve the problem in practice. Therefore, it is essential for this type of the problem to use the simulated path approach, and we use the simulated path model in this paper.

2.2 Setting
We attach a superscript \((i)\) to the random (path dependent) parameters in order to formulate the model in the simulated path approach.

2.2.1 Household
We define a household as a group composed of a householder and members of family in this paper. Wealth at time \(t\) held by a household can be divided into two kinds of wealth: financial wealth \(W^{(i)}_{1,t}\) and non-financial wealth \(W^{(i)}_{2,t}\). The household is exposed to risk associated with two kinds of accidents: the death of the householder and the fire of the house. It is assumed that the death of the householder makes wage earnings stop, and the fire of the house damages a fraction \(\alpha\) of non-financial wealth. The household can purchase the life insurance and the fire insurance to hedge risk in addition to the investment in securities such as stocks and bonds.

2.2.2 Incomes
Income at time \(t\) is the householder’s wage \(m_t\) if the householder is alive and investment returns from financial wealth \(W^{(i)}_{1,t}\). If the householder dies, the household cannot get wages, but draw the survivor’s pension. The amount of the survivor’s pension is calculated based on the wage level. Let \(a^{(i)}_{t_m}\) be the amount of the survivor’s pension because the amount of the survivors’ employees’ pension is dependent on the time of the householder’s death \(t_m\). The income by the wage or the survivor’s pension \(M^{(i)}_{t}\) can be shown as follows:

\[
M^{(i)}_{t} = \tau^{(i)}_{3,t} m^{(i)}_{t} + \left(1 - \tau^{(i)}_{3,t}\right) a^{(i)}_{t_m}
\]

where \(\tau^{(i)}_{3,t}\) are one if a householder is alive on the path \(i\) at time \(t\) and zero if otherwise.

2.2.3 Consumption expenses
There assumes to be two kinds of expenses: the living expenses \(C^{(i)}_{1,t}\) and the purchase of non-financial assets \(C^{(i)}_{2,t}\), such as a house, goods, and repair costs. We need to pay the restoration cost if the fire of the house occurs.

1) Expenses for purchasing a house

\(^2\)Hibiki (2001c, 2003) developed the hybrid model, which not only describes the uncertainties on the simulated path structure but also makes conditional decisions on the tree structure. The hybrid model is allowed to expand the decision space and to make conditional decisions as in the scenario tree model. The simulated path model is a special version of the hybrid model. The formulation and numerical tests with the hybrid model are our future research.
We assume that a household purchases a house with a down payment and a debt loan from banks ($H_t$). Let $t_e$ be the time when the house is purchased. The debt loan $H_t$ is the difference between the price of the house and the down payment. We explain the relationship between a cash flow of purchasing a house and the change of non-financial wealth. The debt loan $H_t$ is a cash inflow, and the consumption expenditures for non-financial asset $C_{2,t}$ is a cash outflow. However, the non-financial wealth, $W_{2,t}$, is increased by expenditures $C_{2,t}$ at time $t_e$. The household has to pay the debt loan periodically under the determined mortgage interest rate and the loan period after the time $t_e + 1$. In this paper, we include periodic payments $C_{1,t}$ in the expenditures for life ($C_{1,t}^{(i)}$).

(2) Restoration cost due to the fire

It is assumed that the fraction $\alpha$ of non-financial wealth $W_{2,t}$ is damaged and the following restoration cost $A_t^{(i)}$ is paid if the fire of the house occurs $^3$.

$$A_t^{(i)} = \tau_{2,t}^{(i)} W_{2,t-1} \alpha (1 - \gamma)$$

where $\tau_{2,t}^{(i)}$ are one if the fire of the house occurs and zero if it does not occur, and $\gamma$ is depreciation ratio of non-financial wealth. $A_t^{(i)}$ does not affect the non-financial wealth. Instead, it affects the cash flow as shown in Equation (7) and (8) in Section 3.

(3) Expenditures for living $C_{1,t}^{(i)}$

We show three kinds of parameters as follows.

$C_{1,t}^{1(i)}$: costs independent of the householder’s death, such as education costs and rent

$C_{1,t}^{2(i)}$: annual payments for mortgage loan (when the householder is alive).

$C_{1,t}^{3(i)}$: other costs for living except $C_{1,t}^{1(i)}$ and $C_{1,t}^{2(i)}$ which are supposed when householder is alive.

Next, we explain how to compute the annual payments for the debt loan and other expenditures for living dependent on the householder’s death.

1. Mortgage loan

   As mentioned earlier, $\tau_{3,t}^{(i)}$ are one if a householder is alive and zero if otherwise. If the household purchases the group credit insurance $^4$, the loan payment is forgiven after the householder was dead. This shows that the annual payments can be $\tau_{3,t}^{(i)} C_{1,t}^{2(i)}$. However, the loan payment is not forgiven if the household purchases the house after the householder was dead. By using the condition that $\tau_{3,t}^{(i)} = 0$ for $t > t_e$ if $\tau_{3,t_e}^{(i)} = 0$, the annual payments for mortgage loan can be

$$\left( 1 - \tau_{3,t_e}^{(i)} + \tau_{3,t}^{(i)} \right) C_{1,t}^{2(i)}.$$ 

2. Change of the consumption level

The household will think it cannot keep the consumption level if the householder dies. It is assumed that the household can keep the normal consumption level if the householder is alive, however the consumption level must be $\kappa$ times the normal level if the householder is dead, where $\kappa$ is the parameter associated with the consumption level. For example, we set $\kappa = 1$ when the household keeps the normal level, and we set $\kappa = 0.7$ when it has to allow for the 70%

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$^3$Non-financial wealth is decreased by $A_t^{(i)}$ due to the fire, but it is increased by $A_t^{(i)}$ by spending the same money to recover the loss.

$^4$Insurance fee for the mortgage loan from financial institutions in the private sector is included in the mortgage interest rate. The group credit insurance needs to be purchased for the mortgage loan from Government Housing Loan Corporation.
consumption level. Therefore, other costs for living except $C_{1,t}$ and $\left(1 - \tau_{3,t}^{(i)} + \tau_{3,t}^{(i)}\right)C_{1,t}^2$ are 5

$$\left\{\tau_{3,t}^{(i)} + \left(1 - \tau_{3,t}^{(i)}\right)\kappa\right\}C_{1,t}^3 = \left\{\kappa + (1 - \kappa)\tau_{3,t}^{(i)}\right\}C_{1,t}^3.$$

The expenditures for living are calculated in total as follows.

$$C_{1,t} = C_{1,t} + \left(1 - \tau_{3,t}^{(i)} + \tau_{3,t}^{(i)}\right)C_{1,t}^2 + \left\{\kappa + (1 - \kappa)\tau_{3,t}^{(i)}\right\}C_{1,t}^3$$

\subsection{2.2.4 Securities}

Investment in risky assets contributes to a hedge against inflation. We invest in $n$ risky assets and cash. A rate of return $R_{jt}$ of risky asset $j$ at time $t$ is calculated using the price $\rho_{jt}$ as follows.

$$R_{jt} = \frac{\rho_{jt}}{\rho_{jt-1}} - 1, \quad (t = 1, \ldots, T)$$

A risk-free rate $r_t$ at time $t (= 0, 1, \ldots, T - 1)$ is fixed in the period from time $t$ to time $t + 1$. We can assume any probability distributions of $R_{jt}$ and $r_t$ in the simulated path approach if we can sample random paths for $R_{jt}$ and $r_t$. However, it is assumed that $R$ is normally distributed with the mean vector $\mu$, and the covariance matrix $\Sigma$ ($R \sim N(\mu, \Sigma)$, and $r_t$ is constant for all $t$. We calculate the price $\rho_{jt}$ by using $R_{jt}$.

\subsection{2.2.5 Life insurance}

We use the term life insurance with maturity $T$ against the householder’s death. If a householder purchases the term life insurance and dies until time $T$, the household can receive the insurance money. In this model, we look upon the life insurance as the financial product which can hedge risk associated with the wage income earned by the householder. When the insurance policy is designed, pure premium should be determined so that the present value of future premium income can be equal to the present value of future premium payments. It is called the principle of equalization of income and expenditure. The yield used in calculating the present value is called the guaranteed interest rate.

Using the principle of equalization of income and expenditure, the relationship between a unit of present value of premium income and the corresponding insurance money $\theta_1$ is shown as:

$$1 = \sum_{t=1}^{T} \frac{\theta_1 \lambda_{1,t}}{(1 + g_1)^t}, \quad \text{or} \quad \theta_1 = \left(\sum_{t=1}^{T} \frac{\lambda_{1,t}}{(1 + g_1)^t}\right)^{-1}$$

where $g_1$ is the guaranteed interest rate of the insurance against death with maturity $T$, and $\lambda_{1,t}$ is the mortality rate at time $t$, or the probability that the person who is alive at time $0$ will die at time $t$.

We can select single payment or level payment when we pay the life insurance premium. Premium of single payment per unit $y_{f1}$ is equal to a unit of the present value of future premium income.

\footnote{If the householder is alive, the consumption level is $\tau_{3,t}^{(i)}$ times the normal level because $\tau_{3,t}^{(i)} = 1$ and $\left(1 - \tau_{3,t}^{(i)}\right)\kappa = 0$. If the householder is dead, the consumption level is $\kappa$ times the normal level because $\tau_{3,t}^{(i)} = 0$ and $\left(1 - \tau_{3,t}^{(i)}\right)\kappa = \kappa$.}
\[ y_{f_1} = 1 \]  
(3)

The premium of level payment per unit \( y_{f_2} \) is calculated as follows, because only insured person who is alive pays the premium.

\[
y_{f_2} = \left[ \sum_{t=0}^{T-1} \left( \frac{1 - \sum_{i=0}^{t} \lambda_{1,i}}{(1 + g_1)^t} \right) \right]^{-1}
\]  
(4)

### 2.2.6 Fire insurance

The household purchases one year fire insurance to hedge the damage of non-financial wealth due to the fire. The household can update the insurance contract every year, and purchase the fire insurance policy corresponding to the future non-financial wealth. Using the principle of equalization of income and expenditure, the relationship between one unit of present value of premium income and the corresponding insurance money \( \theta_2 \) is shown as:

\[
1 = \frac{\theta_2 \lambda_2}{1 + g_2}, \quad \text{or} \quad \theta_2 = \frac{1 + g_2}{\lambda_2}
\]  
(5)

where \( g_2 \) is the guaranteed interest rate of the one year fire insurance, and \( \lambda_2 \) is the rate of the fire, or the probability that the fire occurs. It is independent on time \( t \).

We can only select single payment because of the one year fire insurance. The premium of single payment per unit \( y_F \) is equal to a unit of the present value of future premium income.

\[ y_F = 1 \]  
(6)

### 3 Multi-period ALM optimization model for a household

We formulate the multi-period optimization model in the simulated path approach. We assume that the current time is \( 0(t = 0) \), and a householder retires at time \( T \), which is a planning horizon. As mentioned in Section 2, we invest in \( n \) risky assets and cash, and we can rebalance the positions at each time. We purchase \( T \)-years life insurance at time 0, and one-year fire insurance which is updated every year in the planning period. We can select single payment life insurance or level payment life insurance.

#### 3.1 Notations

1. **Subscript/Superscript**
   - \( j \) : asset \((j = 1, \ldots, n)\).
   - \( t \) : time \((t = 1, \ldots, T)\).
   - \( i \) : path \((i = 1, \ldots, I)\).

2. **Parameters**
   - \( I \) : number of simulated paths.
   - \( \rho_{j0} \) : price of risky asset \( j \) at time 0, \((j = 1, \ldots, n)\).
\[ \rho_{jt}^{(i)} : \text{price of risky asset } j \text{ of path } i \text{ at time } t, \quad (j = 1, \ldots, n; \ t = 1, \ldots, T; \ i = 1, \ldots, I). \]

\[ \rho_{jt}^{(i)} = (1 + R_{jt}^{(i)}) \rho_{j0}, \quad (j = 1, \ldots, n; \ i = 1, \ldots, I) \]

\[ \rho_{jt}^{(i)} = (1 + R_{jt}^{(i)}) \rho_{jt-1}^{(i)}, \quad (j = 1, \ldots, n; \ t = 2, \ldots, T; \ i = 1, \ldots, I) \]

where \( R_{jt}^{(i)} \) is the rate of return of risky asset \( j \) on path \( i \) at time \( t \).

\[ r_0 \] : interest rate in period 1, (the rate at time 0).

\[ r_{i-1}^{(i)} \] : interest rate in period \( t \) (the rate of path \( i \) at time \( t - 1 \)), \( (t = 2, \ldots, T; \ i = 1, \ldots, I). \)

\[ \tau_{1,t}^{(i)} \] : one if a householder dies on path \( i \) at time \( t \) and zero if otherwise.

\[ \tau_{2,t}^{(i)} \] : one if the fire of the house occurs and zero if it does not occur.

\[ \tau_{3,t}^{(i)} \] : one if a householder is alive on path \( i \) at time \( t \) and zero if otherwise.

\[ \lambda_{1,t} \] : mortality rate at time \( t \): \[ \lambda_{1,t} = \Pr(\tau_{1,t} = 1) = \frac{1}{T} \sum_{i=1}^{I} \tau_{1,t}^{(i)} \]

\[ \lambda_2 \] : rate of the fire (which is time independent): \[ \lambda_2 = \Pr(\tau_{2,t} = 1) = \frac{1}{T} \sum_{i=1}^{I} \tau_{2,t}^{(i)} \]

\[ g_1 \] : guaranteed interest rate on life insurance policies.

\[ f_1 \] : one if single payment life insurance is bought and zero if level payment life insurance is bought.

\[ y_{f1} \] : premium of single payment life insurance per unit: \( y_{f1} = 1 \) (A unit of insurance policy corresponds to the present premium of 1 yen.)

\[ y_{f2} \] : premium of level payment life insurance per unit: \( y_{f2} = \left( 1 - \sum_{t=0}^{T-1} \frac{\lambda_{1,t}}{(1 + g_1)^t} \right)^{-1} \)

\[ y_{L,t}^{(i)} \] : premium of life insurance per unit at time \( t \):

\( y_{L,t}^{(i)} = y_{f1} \cdot f_1 \tau_{4,t} + y_{f2} \cdot (1 - f_1) \tau_{3,t}^{(i)} \), where \( \tau_{4,0} = 1 \), \( \tau_{4,t} = 0(t \neq 0) \).

\[ \theta_1 \] : life insurance money per unit: \[ \theta_1 = \left( \sum_{i=1}^{T} \frac{\lambda_{1,t}^{(i)}}{(1 + g_1)^t} \right)^{-1} \]

\[ L_t^{(i)} \] : life insurance money per unit on path \( i \) at time \( t \): \[ L_t^{(i)} = \tau_{1,t}^{(i)} \theta_1 \]

\[ g_2 \] : guaranteed interest rate on fire insurance policies.

\[ y_F \] : premium of fire insurance per unit: \( y_F = 1 \)

\[ \theta_2 \] : one year fire insurance money per unit: \[ \theta_2 = \frac{1 + g_2}{\lambda_2} \]

\[ F_t^{(i)} \] : one year fire insurance money per unit on path \( i \) at time \( t \): \[ F_t^{(i)} = \tau_{2,t}^{(i)} \theta_2 \]

\[ \alpha \] : loss ratio of non-financial wealth due to the fire of the house.

\[ A_t^{(i)} \] : loss of non-financial wealth due to the fire of the house on path \( i \) at time \( t \):

\[ A_t^{(i)} = \tau_{2,t}^{(i)} W_{2,t-1} \alpha (1 - \gamma) \]

where \( \gamma \) is depreciation ratio of non-financial asset.

\[ M_t^{(i)} \] : wage income a householder earns or survivor's pension a household receives on path \( i \) at time \( t \):

\[ H_t^{(i)} \] : debt loan on path \( i \) at time \( t \).
$C_t^{(i)}$: total consumption expenditures on path $i$ at time $t$.

$W_{1,t}^{(i)}$: financial wealth on path $i$ at time $t$. ($W_{1,0}$ is an initial financial wealth at time 0.)

$W_{2,t}^{(i)}$: non-financial wealth on path $i$ at time $t$: \( W_{2,t}^{(i)} = (1 - \gamma)W_{2,t-1}^{(i)} + C_{2,t}^{(i)} \) ($W_{2,0}$ is an initial non-financial wealth at time 0.)

$W_E$: lower bound of expected terminal financial wealth

$\beta$: Probability level used in the CVaR calculation.

$L_{v,t}$: lower bound of cash at time $t$. When $L_{v,t} < 0$, the borrowing can be allowed.

### 3.2 Objective function, return requirement and cash flow except trading asset

#### (1) Objective function

The objective is the maximization of the CVaR associated with terminal financial wealth subject to the minimum return requirement, as follows.

$$\text{CVaR}_\beta = \max \left\{ V_\beta - \frac{1}{(1 - \beta)I} \sum_{i=1}^{I} q(i) \left| W_{1,T}^{(i)} - V_\beta + q(i) \geq 0, (i = 1, \ldots, I) \right. \right\}$$

Even if CVaR of $W_0 - W_{1,T}^{(i)}$ is used to minimize the objective, we have the same solutions as the solutions derived from the maximization of CVaR of $W_{1,T}^{(i)}$.

#### (2) Return requirement

We define the expected terminal financial wealth $E[W_{1,T}]$ as a return measure. The lower bound is $W_E$, and therefore the minimum return requirement is formulated as:

$$\sum_{i=1}^{I} W_{1,T}^{(i)} \geq W_E$$

#### (3) Cash flow except trading assets

Cash flow constraints are important in the multi-period optimization approach. Cash flow except trading assets $D_t^{(i)}$ is associated with income, expenditures, and insurance. It is formulated as follows.

1. $t = 1, \ldots, T - 1$

\[
D_t^{(i)} = M_t^{(i)} + H_t^{(i)} - C_t^{(i)} - y_{L,t}^{(i)}u_L - y_{F,t}^{(i)}u_{F,t} + L_t^{(i)}u_L + F_t^{(i)}u_{F,t-1} - A_t^{(i)},
\]

\[
(t = 1, \ldots, T - 1)
\]


2. $t = T$: Insurance payment is not required at time $T$.

\[ D_T^{(i)} = M_T^{(i)} + H_T^{(i)} - C_T^{(i)} + L_T u_L + F_T u_{F,T-1} - A_T^{(i)} \]  

(8)

3.3 Formulation

\[ \text{Maximize} \quad V_\beta - \frac{1}{(1-\beta)T} \sum_{i=1}^{I} q^{(i)} \]  

(9)

subject to

\[ \sum_{j=1}^{n} \rho_{j0} z_{j0} + v_0 + y_{L,0} u_L + y_{F} u_{F,0} = W_{1,0} \]  

(10)

\[ (W_{1,1}^{(i)} =) \sum_{j=1}^{n} \rho_{j1} z_{j0} + (1 + r_0) v_0 + D_1^{(i)} = \sum_{j=1}^{n} \rho_{j1} z_{j1} + v_1^{(i)}, \quad (i = 1, \ldots, I) \]  

(11)

\[ (W_{1,t}^{(i)} =) \sum_{j=1}^{n} \rho_{jt} z_{jt-1} + (1 + r_{t-1}^{(i)}) v_{t-1}^{(i)} + D_t^{(i)} = \sum_{j=1}^{n} \rho_{jt} z_{jt} + v_{t}^{(i)}, \quad (t = 2, \ldots, T-1; \ i = 1, \ldots, I) \]  

(12)

\[ W_{1,T}^{(i)} = \left\{ \sum_{j=1}^{n} \rho_{jT} z_{jT-1} + (1 + r_{T-1}^{(i)}) v_{T-1}^{(i)} \right\} + D_T^{(i)}, \quad (i = 1, \ldots, I) \]  

(13)

\[ \frac{1}{T} \sum_{i=1}^{I} W_{1,T}^{(i)} \geq W_E \]  

(14)

\[ W_{1,T}^{(i)} - V_\beta + q^{(i)} \geq 0, \quad (i = 1, \ldots, I) \]  

(15)

\[ z_{jt} \geq 0, \quad (j = 1, \ldots, n; \ t = 0, \ldots, T-1) \]  

\[ v_0 \geq 0 \]  

\[ v_{t}^{(i)} \geq L_{u,t}, \quad (t = 1, \ldots, T-1; \ i = 1, \ldots, I) \]  

\[ u_L \geq 0 \]  

\[ u_{F,t} \geq 0, \quad (t = 0, \ldots, T-1) \]  

\[ q^{(i)} \geq 0, \quad (i = 1, \ldots, I) \]  

\[ V_\beta : \text{free} \]

4 Numerical examples

4.1 Preparation

We test numerical examples using the parameters (Household B in Section 6) in Hibiki, Komoriibayashi, and Toyoda[2005]. A householder is 30 years old and a spouse is 28 years old. The first child is 0 years old, and the second child will be born in three years. The householder works at a financial institution, and the household plans that it will prepare 20 million yen as a down payment eleven years later and buy an apartment in the center of Tokyo which costs 50
million yen. 20 million yen is paid at the time \( t_e = 11 \) when the house is bought. We borrow 30 million yen and the mortgage loan is equally paid over 20 years. Equal yearly payment is calculated with the mortgage investment rate(6\%) \(^6\). The parents make an educational plan that the children will go to the private elementary school, junior high school, high school, and university. The parameter values used in the examples are shown in Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of risky assets</td>
<td>( n = 1 )</td>
</tr>
<tr>
<td>length of one period</td>
<td>one year</td>
</tr>
<tr>
<td>retirement age of a householder</td>
<td>60 years old</td>
</tr>
<tr>
<td>number of periods</td>
<td>( T = 30 )</td>
</tr>
<tr>
<td>expected rate of return of a risky asset</td>
<td>( \mu = 0.1 )</td>
</tr>
<tr>
<td>standard deviation of rate of return of a risky asset</td>
<td>( \sigma = 0.2 )</td>
</tr>
<tr>
<td>risk-free rate</td>
<td>( r = 0.04 )</td>
</tr>
<tr>
<td>mortality rate</td>
<td>( \lambda_{1,t}(\dagger) )</td>
</tr>
<tr>
<td>rate of the fire</td>
<td>( \lambda_0 = 0.005 )</td>
</tr>
<tr>
<td>guaranteed rate on life insurance</td>
<td>( g_1 = 0.05 )</td>
</tr>
<tr>
<td>guaranteed rate on fire insurance</td>
<td>( g_2 = 0.05 )</td>
</tr>
<tr>
<td>payment of life insurance</td>
<td>level payment: ( f_1 = 0 )</td>
</tr>
<tr>
<td>initial financial wealth (million yen)</td>
<td>( W_{1,0} = 10 )</td>
</tr>
<tr>
<td>initial non-financial wealth (million yen)</td>
<td>( W_{2,0} = 10 )</td>
</tr>
<tr>
<td>depreciation rate of non-financial wealth</td>
<td>( \gamma = 0.03 )</td>
</tr>
<tr>
<td>loss of non-financial wealth due to the fire</td>
<td>( \alpha = 1 )</td>
</tr>
<tr>
<td>lower bound of cash (million yen)</td>
<td>( L_{v,0} = 0, L_{v,t} = -10(t \neq 0) )</td>
</tr>
<tr>
<td>lower bound of expected terminal financial asset(million yen)</td>
<td>( W_E = 52.616 )</td>
</tr>
<tr>
<td>probability level</td>
<td>( \beta = 0.8 )</td>
</tr>
<tr>
<td>number of paths</td>
<td>( I = 5,000 )</td>
</tr>
</tbody>
</table>

\( \dagger \) The rates are estimated by the “life insurance standard life table 1996 for men.

The wage income depends on the householder’s age and his occupation. We calculate the wage income of the household over time based on the Census of wage by Ministry of Health, Labor and Welfare(2003). The household whose householder is a financial institution employee has a highly increasing rate in salary until 50 years old, however the wage income declines afterwards. The consumption expenditure depends on the wage income, family structure and school(education) plan. We calculate average consumption expenditures with respect to each number of family and each income level of family based on the national survey of family income and expenditure(1999) by Statistic Bureau, Ministry of Internal Affairs and Communications. We calculate average educational expenses based on the survey of household expenditure on education per student(2001), the survey of student life by Ministry of Education, Culture, Sports, Science and Technology.

\(^6\)The annuity is 2.616 million yen. We have 19 payments from \( t = 12 \) to \( t = 30 \), and one payment after retirement. It is assumed that the necessary terminal financial wealth is 70 million yen and the severance pay is 20 million yen. Therefore the lower bound of the expected terminal financial wealth \( (W_E) \) is 52.616 million yen.
We test the following 20 combinations to clarify the effects of three factors: ① receiving the survivor’s pension, ② exemption from the mortgage loan, ③ change of the consumption level.

<table>
<thead>
<tr>
<th>Factor</th>
<th>Yes</th>
<th>No</th>
<th>Cases</th>
</tr>
</thead>
<tbody>
<tr>
<td>receiving the survivor’s pension</td>
<td>$p_f = 1$</td>
<td>$p_f = 0$</td>
<td>2 cases</td>
</tr>
<tr>
<td>exemption from the mortgage loan</td>
<td>$n_p = 1$</td>
<td>$n_p = 0$</td>
<td>2 cases</td>
</tr>
<tr>
<td>change of the consumption level</td>
<td>$\kappa = 0.6 \sim 1.0$ (by 0.1)</td>
<td>5 cases</td>
<td></td>
</tr>
</tbody>
</table>

The model in Hibiki, Komoribayashi, and Toyoda [2005] corresponds to the model with $p_f = 0$, $n_p = 0$, $\kappa = 1$.

### 4.2 Result

![Figure 2: CVaR and life insurance money](image)

Figure 2 shows the CVaR on the left-hand side and life insurance money on the right-hand side. When the household keeps the consumption level lower – we solve the problem with lower $\kappa$ —, the CVaR is higher and the household needs the lower life insurance money. This is because the receipt of the life insurance money and the reduction in consumption can make up for the loss of wage income caused by the householder’s death. If the household receives the survivor’s pension ($p_f = 1$), the life insurance money can be bought lower by about 30 million yen for the same reason.

Exemption from mortgage payment slightly improves the objective function value. However it does not affect the objective function value and life insurance money. The reason is that we purchase the life insurance at time 0, but the house is bought at time 11, and therefore the loan payment is not forgiven if the house is bought after the householder was dead. However, this does not show that exemption from mortgage payment is not effective. Figure 3 shows the conditional expected terminal financial wealth at the time of the householder’s death. The value at time 0 shows the expected value under the condition that the householder does not die in the
planning period. If the householder dies after time 12, the loan payment is forgiven. Therefore, the expected terminal financial wealth after time 12 for \( np = 1 \) are larger than those for \( np = 0 \) because the reduction in loan payment contributes to the increase in the terminal financial wealth. The effect of the reduction in loan payment fades as the time of the householder’s death becomes late, and the difference between the value for \( np = 1 \) and the value for \( np = 0 \) diminishes.

The expected terminal financial wealth is increasing as the time of the householder’s death becomes late. The reason is that the household gets the wage income in the longer period before the householder dies, and the life insurance money when the householder dies. The expected wealth when the householder is alive is smaller than the expected wealth when the householder dies after time 14. If the householder dies earlier, especially before buying a house, the householder receives the life insurance money. However, the expected terminal wealth tends to be lower because of the lower survivor’s pension.

The left-hand side of Figure 4 shows investment units of the risky asset for five kinds of \( \kappa \) value, \( pf = 1 \), and \( np = 1 \). The trend of the optimal investment units at each time is not dependent on \( \kappa \) value, however the lower \( \kappa \) value is, the smaller the investment unit is. The reason is that terminal financial wealth increases by reducing the consumption level, and therefore we do not have to take risk by investing in a risky asset. We have the same characteristics for the combination of both \( pf \) and \( np \). The right-hand side of Figure 4 shows investment units

---

The reader should pay attention to looking at Figure 3.

The conditional expected terminal wealth at time 1 is large because the conditional expected terminal price of a risky asset is high. This is a sampling error because there are four paths at time 1 when the householder dies.
of a risky asset for four combinations of both \( pf \) and \( np \), and \( \kappa = 1 \). When we consider the receipt of survivor’s pension and exemption from the loan payment, we can hedge risk against the householder’s death and expect the increase in the terminal financial wealth. Therefore the investment units of a risky asset are reduced.

![Figure 4: Investment unit of the risky asset](image)

![Figure 5: Average cash](image)

Figure 5 shows the average cash for five kinds of \( \kappa \) value, \( pf = 1 \), and \( np = 1 \) on the left-hand side, and for four combinations of both \( pf \) and \( np \), and \( \kappa = 1 \) on the right-hand side. The average cash increases gradually until time 10, however it decreases drastically at time 11 because of purchasing a house. After time 12, it increases slightly over years, however it increases drastically after time 25 because the second child graduates from university and we do not have to pay the education cost. When \( \kappa \) value is small, the household receives the survivor’s pension, or the loan payment is forgiven, average cash increases because it expects to increase the terminal financial wealth, and it reduces the investment in a risky asset.

Figure 6 shows the investment ratio of a risky asset for five kinds of \( \kappa \) value, \( pf = 1 \), and \( np = 1 \) on the left-hand side, and for four combinations of both \( pf \) and \( np \), and \( \kappa = 1 \) on the right-hand side. The average cash increases much more than the investment unit of a risky asset.
until the time the house is bought. On the other hand, the average investment ratio of a risky asset is reduced slightly because the average price of a risky asset rises with the 10% expected rate of return.

Though the investment unit of a risky asset does not change at time 11, the average investment ratio of a risky asset increases drastically because the house is bought and a big cash outflow occurs to make a down payment. The average ratio of a risky asset increases gradually after the time. However, it goes down as the average cash goes up. The result is summarized as follows. When the survivor’s pension is received, the loan payment is forgiven, and the consumption level $\bar{c}$ is small, the cash increases and the average investment ratio of a risky asset decreases.

Figure 7 shows the sampled distributions of the terminal financial wealth derived by the optimal solutions. The right-hand side is a magnified view below the VaR. The tail of the distribution shifts to the right and the downside risk can be decreased by considering the survivor’s pension. This is the result we expect from the CVaR in the left-hand side of Figure 2.
We compare the sampled distribution with the associated normal and log-normal distributions in Figure 8. We derive the sampled distribution in the case where $p_f = 1$, $n_p = 1$, $\kappa = 1$. The normal and log-normal distributions have the same expected value and standard deviation as the sampled distribution. The sampled distribution has the thinner tail than the normal distribution, and it is similar in shape to the log-normal distribution. The reason is as follows. It is assumed that the returns of the risky asset have no time-series correlation, and therefore the terminal price distribution of the risky asset is similar in shape to the log-normal distribution.

Figure 9 shows the optimal fire insurance money for five kinds of $\kappa$ value, $p_f = 1$, and $n_p = 1$ on the left-hand side, and for four combinations of both $p_f$ and $n_p$, and $\kappa = 1$ on the right-hand side. We find that it is not affected by three factors, or receiving the survivor’s pension, exemption from the mortgage loan, and change of the consumption level.

Figure 9: Optimal fire insurance money
5 Sensitivity analysis on home buying

We have four kinds of parameters associated with home buying; time, down payment, loan period, and mortgage interest rate. We solve the problem with the fixed parameters in the examples of Section 4. We analyze the sensitivity of these parameters. We set $W_E = 70$ million yen because the lower bound constraint of the expected terminal financial wealth may not be active for the lower mortgage interest rate. All 36 combinations of three kinds of time ($t_e = 5, 10, 15$ (years)), three kinds of down payments ($10, 20, 30$ million yen), and four kinds of mortgage interest rates ($3\%, 4\%, 5\%, 6\%$) are solved for $pf = 1$, $np = 1$, and $\kappa = 1$. The loan period depends on the time when the house is bought, and is set from the home buying time to the retirement ($T - t_e = 25, 20, 15$ (years)).

Figure 10 consists of three figures. Each figure shows the CVaR(objective function value) for each home buying time. If the mortgage interest rate is equal to the risk-free rate(4%), the CVaR is not dependent on the down payment. If the mortgage interest rate is lower than the risk-free rate, the smaller the down payment is, the larger the CVaR is. This reason is that it is better to take advantage of the investment in more cash with the higher risk-free rate instead of making more down payments, and the household can obtain the difference between two interests.

When the home buying time becomes later, the objective function values get less affected by down payment and mortgage interest rate.

Figure 10: Objective function values (1)

Figure 10 is transformed into Figure 11 to clarify the relationship between the CVaR and the combinations of the home buying time and the mortgage interest rate. Figure 11 consists of three figures, which show the CVaR for each down payment. When the mortgage interest rate is low, and we make the same down payment, the earlier the home buying time is, the higher the CVaR is. The reason is that the household does not have to pay the rent instead of paying the interest rate if the home buying time is earlier. If the mortgage interest rate is higher, the household has to make more loan payment, and the later home buying time is superior to the earlier one.

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9We cannot obtain the feasible solution for the combination of time 5 of home buying time and 30 million yen of down payment. This reason is that we cannot make much down payment and purchase a house earlier. If the lower bound of cash $L_{e, t}$ goes down, we will obtain the feasible and optimal solution.
Figure 11: Objective function values (2)

Figure 12 consists of three figures, and shows the optimal life insurance money for each home buying time. The higher the mortgage interest rate is, the more the optimal life insurance money is. The reason is that the higher loan payment due to the higher mortgage interest rate reduces more financial wealth, and the household has to cover the loss by the life insurance money if the householder is dead. On the other hand, the household needs to make low down payment for the low mortgage interest rate, and it needs to make high down payment for the high mortgage interest rate in order to make the optimal life insurance money low. If we make the lower down payment and the higher loan payment, and the householder dies earlier, it can keep more financial wealth in order to forgive the loan payment. However, the financial wealth is reduced due to the higher loan payment. Therefore, when the mortgage interest rate is lower, it can keep more financial wealth for the lower down payment. However, when the mortgage interest rate is higher, it keeps less financial wealth, and it has to purchase more life insurance to cover the loss. When the down payment is high, the financial wealth is not affected by the mortgage interest rate.

Figure 12: Optimal life insurance money (1)

Figure 12 is transformed into Figure 13 to clarify the relationship between the life insurance
money and the combinations of the home buying time and the mortgage interest rates. Figure 13 consists of three figures, which show the life insurance money for each down payment. When the mortgage interest rate is low, and the household makes the same down payment, the earlier the home buying time is, the lower the life insurance money is. However, when the mortgage interest rate is high, the later the home buying time is, the lower the life insurance money is. The reason is that the earlier the household purchases the house, the higher the probability the loan payment is forgiven due to the householder’s death. However, when the interest rate is high, the interest payment is larger than the exemption from the loan payment, and we have the opposite result. When the home buying time is late, the financial wealth is not affected by the mortgage interest rate.

![Figure 13: Optimal life insurance money (2)](image1)

Figure 14 shows the optimal investment units of a risky asset when the household purchases the house at time 10. When the mortgage interest rate is high, the household needs to invest in a risky asset and to increase the expected terminal financial wealth because the household has to make the high loan payment. The more the down payment is, the smaller the difference among the investment units because the result is not affected by the mortgage interest rate. Due to space limitation, we omit the case for $t_e = 5$ and $t_e = 15$.

![Figure 14: Optimal investment unit of the risky asset ($t_e = 10$)](image2)
6 Formulation using a grouped path

The problem size depends on the number of the planning period \((T)\), the number of paths \((I)\), and the number of risky assets \((n)\). The number of decision variables and the number of constraints except upper and lower bound constraints and non-negativity constraints, as follows.

Number of decision variables : \(T(n + I + 1) + 3\)

Number of constraints : \(TI + 2\)

Both numbers are almost \(TI\). The number of paths \((I)\) is larger than the number of period \((T)\) in the simulated path approach. We need a lot of paths to describe the mortality rate associated with life insurance and the rate of the fire associated with fire insurance. According to the life insurance standard life table(1996), the mortality rate for 40 years old men is 0.156%. Even if the householder dies on a path to describe the appropriate mortality rate, we need 641 paths. Therefore, we need to increase the number of paths to improve the accuracy of description of uncertainties, but the problem size gets large.

On the other hand, the reason we need a lot of sample paths is that we describe the uncertainties accurately and calculate an appropriate value of risk measure. We do not use the paths where the terminal wealth are larger than the VaR if we use the CVaR as the risk measure \(^{10}\). We aggregate the paths we may not use to calculate the CVaR, and we think of the aggregated paths as a virtual path with the probability \(\frac{|G|}{j}G\) where \(G\) is the set of the aggregated paths and \(|G|\) is the number of the paths. We call a virtual path ‘grouped path’ and we formulate the model with the grouped path.

Figure 15 shows a sketch of setting the grouped path.

![Figure 15: Setting the grouped path](image)

The number of paths is twelve. Three paths we may use to calculate the CVaR are \(i = 10, 11, 12\), and the set \(G\) consists of nine paths. The more the paths included in the set \(G\) are, the smaller the problem size is. However, there is a possibility of the set \(G\) which has the necessary paths to calculate the CVaR. There is a tradeoff between the problem size(or computation time) and the accuracy, and determining the set \(G\) depends on the aspect of the

\(^{10}\)We can say this idea for downside risk measure such as the LPM(lower partial moments).
problem.
At first, we formulate the model under a given set \( G \). Second, we propose the method to determine the set \( G \), and we test the method with numerical examples.

### 6.1 Formulation

Constraints associated with the paths in the set \( G \) are aggregated, and the aggregated constraints are described such as Equations (19), (20), (22). If the paths are not in the set \( G \), we describe the same constraints, such as Equations (18), (21), (23).

**Maximize** \[ V_\beta = \frac{1}{(1-\beta)I} \sum_{i \in G} q^{(i)} \] (16)

**subject to**

\[
\sum_{j=1}^{n} \rho_{j0} z_{j0} + \nu_0 + y_{L,0} u_L + y_{F,0} u_F = W_{1,0}
\] (17)

\[
\sum_{j=1}^{n} \rho_{ji}^{G} z_{j0} + (1 + r_0) \nu_0 + D_1^{G} = \sum_{j=1}^{n} \rho_{ji}^{G} z_{j1} + \frac{1}{|G|} \sum_{i \in G} v_1^{(i)}
\] (18)

\[
\sum_{j=1}^{n} \rho_{ji}^{(i)} z_{j0} + (1 + r_0) \nu_0 + D_1^{(i)} = \sum_{j=1}^{n} \rho_{ji}^{(i)} z_{j1} + v_1^{(i)} , \quad (i \not\in G)
\] (19)

\[
\sum_{j=1}^{n} \rho_{ji}^{G} z_{j,t-1} + \frac{1}{|G|} \sum_{i \in G} \left(1 + r_{i-1}^{(i)}\right) v_{t-1}^{(i)} + D_t^{G} = \sum_{j=1}^{n} \rho_{ji}^{G} z_{jt} + \frac{1}{|G|} \sum_{i \in G} v_t^{(i)}, \quad (t = 2, \ldots, T-1)
\] (20)

\[
\sum_{j=1}^{n} \rho_{jt}^{(i)} z_{j,t-1} + \left(1 + r_{t-1}^{(i)}\right) v_{t-1}^{(i)} + D_t^{(i)} = \sum_{j=1}^{n} \rho_{jt}^{(i)} z_{jt} + v_t^{(i)} , \quad (t = 2, \ldots, T-1; \ i \not\in G)
\] (21)

\[W_{1,T}^G = \left\{ \sum_{j=1}^{n} \rho_{jt}^{G} z_{j,T-1} + \frac{1}{|G|} \sum_{i \in G} \left(1 + r_{T-1}^{(i)}\right) v_{T-1}^{(i)} \right\} + D_T^G
\] (22)

\[W_{1,T}^{(i)} = \left\{ \sum_{j=1}^{n} \rho_{jt}^{(i)} z_{j,T-1} + \left(1 + r_{T-1}^{(i)}\right) v_{T-1}^{(i)} \right\} + D_T^{(i)} , \quad (i \not\in G)
\] (23)

\[\frac{1}{T} \left( |G| W_{1,T}^G + \sum_{i \in G} W_{1,T}^{(i)} \right) \geq W_E
\] (24)

\[W_{1,T}^{(i)} - V_\beta + q^{(i)} \geq 0, \quad (i \not\in G)
\] (25)

\[z_{jt} \geq 0, \quad (j = 1, \ldots, n; \ t = 0, \ldots, T-1)
\]

\[v_0 \geq 0
\]

\[v_t^{(i)} \geq L_{v,t}, \quad (t = 1, \ldots, T-1; \ i = 1, \ldots, I)
\] (26)

\[u_L \geq 0
\]

\[u_{F,t} \geq 0, \quad (t = 0, \ldots, T-1)
\]
\( q^{(i)} \geq 0, \quad (i \notin G) \)

\[ V_\beta : \text{free} \]

where

\[
\rho^{(i)}_{jt} = \frac{1}{|G|} \sum_{i \in G} \rho^{(i)}_{jt}
\]

\[ D^G_t = M^G_t + H^G_t - C^G_t - y^G_{L,t}u_L - y^G_{F,t}u_F + \tau^G_{1,t}u_{1L} + \tau^G_{2,t}u_{2F,t-1} - \frac{1}{|G|} \sum_{i \in G} \tau^{(i)}_{2,t}(1-\gamma)W_{2,t-1}\alpha, \quad (t = 1, \ldots, T - 1) \]

\[ D^G_t = M^G_t + H^G_t - C^G_t - \tau^G_{1,t}u_{1L} + \tau^G_{2,t}u_{2F,T-1} - \frac{1}{|G|} \sum_{i \in G} \tau^{(i)}_{2,T}(1-\gamma)W_{2,T-1}\alpha \]

\[ M^G_t = \frac{1}{|G|} \sum_{i \in G} M^{(i)}_t, \quad H^G_t = \frac{1}{|G|} \sum_{i \in G} H^{(i)}_t, \quad C^G_t = \frac{1}{|G|} \sum_{i \in G} C^{(i)}_t \]

\[ \tau^G_{1,t} = \frac{1}{|G|} \sum_{i \in G} \tau^{(i)}_{1,t}, \quad \tau^G_{2,t} = \frac{1}{|G|} \sum_{i \in G} \tau^{(i)}_{2,t}, \quad \tau^G_{3,t} = \frac{1}{|G|} \sum_{i \in G} \tau^{(i)}_{3,t} \]

\[ y^G_{L,t} = y_{f_1} \cdot f_1\tau_{4,t} + y_{f_2} \cdot (1 - f_1)\tau^G_{3,t} \]

If the interest rate is constant \(r_t\) at each time, and we replace Equation (26) for \(i \in G\) with the following Equation (27), we can replace \(|G|\) decision variables \(v^{(i)}_t\) for \(i \in G\) with one decision variable \(v^G_t\), and we reduce the problem size.

\[ v^G_t = \frac{1}{|G|} \sum_{i \in G} v^{(i)}_t \geq L_t \quad (27) \]

In the case, we need to replace Equations (18), (20), (22), (26) with the following equations.

\[
\sum_{j=1}^n \rho^G_{1j}z_{j0} + (1 + r_0)v_0 + D^G_1 = \sum_{j=1}^n \rho^G_{1j}z_{j1} + v^G_1 \quad (28)
\]

\[
\sum_{j=1}^n \rho^G_{jt}z_{j,t-1} + (1 + r_{t-1})v^G_{t-1} + D^G_t = \sum_{j=1}^n \rho^G_{jt}z_{jt} + v^G_t, \quad (t = 2, \ldots, T - 1) \quad (29)
\]

\[
W^G_{1,T} = \left\{ \sum_{j=1}^n \rho^G_{jT}z_{j,T-1} + (1 + r^G_{T-1})v^G_{T-1} \right\} + D^G_T \quad (30)
\]

\[ v^{(i)}_t \geq L_{v,t}, \quad (t = 1, \ldots, T - 1; \ i \notin G) \quad (31) \]

The number of decision variables and the number of constraints except upper and lower bound constraints and non-negativity constraints, as follows.

Number of decision variables : \(T(I - |G| + n + 1) + 3\)

Number of constraints : \(T(I - |G| + 1) + 1\)

In Section 6.3, we make the problems reduced under the condition that the interest rate is constant, and we test numerical examples.
6.2 Setting the set $G$

We examine the optimal solutions derived in Section 4 in order to determine the set $G$. It is assumed that there is a risky asset, and the wage income and consumption expenditures are not random if the householder is not dead or the fire of the house does not occur because the household does not receive the life insurance money, survivor’s pension or the fire insurance money. Therefore the return of a risky asset is a main factor that influences the terminal financial wealth if the householder is alive. The correlation coefficient between the terminal financial wealth and the terminal price of a risky asset for the paths where the householder is alive is 0.987. The value is almost one, and we should include the paths where the terminal price of a risky asset $(\rho_{1,T})$ is high under the condition that the householder is alive in the set $G$. We need to examine what number of the paths is appropriate. We show the relationship between the rank of paths which is sorted by the terminal price of a risky asset and the rank of paths under the VaR in Figure 16. The right-hand side is a magnified view of a part of the left-hand side.

There is highly possible that the terminal financial wealth on the path with the lower price of a risky asset is smaller than the VaR. Paths under the VaR are reduced gradually in the hit exceeding about 1,000 lower ranks, and we do not have the paths under the VaR if the rank of path is over 1,871. This is just the ex-post analysis, but it shows that 2,622 paths with the higher terminal price of a risky asset become beyond the VaR, and they can be included in the set $G$.

![Figure 16: Number of paths under the VaR when paths are sorted by the terminal prices of risky asset](image)

We do not have the wage income instead of receiving the life insurance money and the survivor’s pension when the householder is dead. The terminal financial wealth in the case where the younger householder dies is smaller than the case that the older householder dies because the lower wage income in total and the survivor’s pension are lower. We show the relationship
between the rank of paths which is sorted by the terminal price of a risky asset and the rank of paths under the VaR in Figure 17 as in Figure 16. The right-hand side is a magnified view of a part of the left-hand side.

Figure 17: Relationship between the number of paths under the VaR and time of the householder’s death

There is highly possible that the terminal financial wealth on the path where the householder is death is smaller than the VaR. Paths under the VaR are reduced gradually in the hit exceeding about 40 lower ranks, and we do not have the paths under the VaR if the rank of path is over 60 to 90. This is just the ex-post analysis, but it shows that paths after \( t_m = 10, 11 \) become beyond the VaR, and they can be included in the set \( G_1 \).

We use two kinds of parameters \( k_1, k_2 \), and we define the set \( G \) using the following procedure in consideration of the above-mentioned features. We examine the accuracy of computation for several \( \beta \) values.

1. We include the path \( i \) in the set \( G \) where \( \tau_{1,t}^{(i)} = 1 \) for \( t = (1 - k_2 \beta)T + 1, \ldots, T \). \( |G|_2 \) is the number of paths in the set \( G \).

2. In addition to the above-mentioned paths, we include the \( |G|_1 \) paths in the set \( G \) which have the largest \( |G|_1 \) weighted terminal price (\( \rho_T^{(i)} \)) where \( |G|_1 = k_1 \beta I - |G|_2 \), and

\[ \begin{array}{|c|c|c|c|c|}
\hline
pf & np & \text{# of paths under VaR} & \text{achieved rank} & \text{achieved time} \\
\hline
0 & 0 & 34 & 50 & 10 \\
0 & 1 & 35 & 50 & 10 \\
1 & 0 & 37 & 58 & 11 \\
1 & 1 & 37 & 58 & 11 \\
\hline
\end{array} \]

\[ \text{11We show the number of paths under the VaR, and the associated achieved rank and time, as follows.} \]

\[ \text{12The set \( G \) associated with the larger \( \beta \) value includes more paths, and therefore we propose the procedure which depends on the \( \beta \) value.} \]
\[ \rho^{(i)}_T = \sum_{j=1}^{n} w_j \rho^{(i)}_{j,T}. \]

For simplicity, we set \( w_j = 1 \). \( k_1 \) is the fraction of paths in the set \( G \) to \( \beta I \). Paths in the set \( G \) are the ones which are expected to be removed when the risk measure is calculated.

\( k_1 \beta I \) paths are included in the set \( G \) by the procedure. The case where \( k_1 = 0 \) and \( k_2 = 0 \) corresponds to the original case.

### 6.3 Numerical analysis

#### Table 2: Problem size

| \( \beta \) | \( k_1 \) | \( G \) | \( |G|_1 \) | \( |G|_2 \) | Constraints | Variables |
|-----------|--------|------|--------|--------|------------|-----------|
| 0.80      | 0.60   | 2,400| 2,043  | 357    | 78,031 (52.0%) | 78,092 (52.0%) |
|           | 0.70   | 2,800| 2,443  | 357    | 66,031 (44.0%) | 66,092 (44.0%) * |
|           | 0.75   | 3,000| 2,643  | 357    | 60,031 (40.0%) | 60,092 (40.0%) |
|           | 0.80   | 3,200| 2,843  | 357    | 54,031 (36.0%) | 54,092 (36.0%) * |
|           | 0.85   | 3,400| 3,043  | 357    | 48,031 (32.0%) | 48,092 (32.0%) * |
|           | 0.90   | 3,600| 3,243  | 357    | 42,031 (28.0%) | 42,092 (28.0%) * |
|           | 0.95   | 3,800| 3,443  | 357    | 36,031 (24.0%) | 36,092 (24.1%) |
| 0.85      | 0.60   | 2,550| 2,178  | 372    | 73,531 (49.0%) | 73,592 (49.0%) |
|           | 0.70   | 2,975| 2,603  | 372    | 60,781 (40.5%) | 60,842 (40.5%) |
|           | 0.75   | 3,188| 2,816  | 372    | 54,406 (36.3%) | 54,467 (36.3%) |
|           | 0.80   | 3,400| 3,028  | 372    | 48,031 (32.0%) | 48,092 (32.0%) |
|           | 0.85   | 3,613| 3,241  | 372    | 41,656 (27.8%) | 41,717 (27.8%) * |
|           | 0.90   | 3,825| 3,453  | 372    | 35,281 (23.5%) | 35,342 (23.6%) * |
|           | 0.95   | 4,038| 3,666  | 372    | 28,906 (19.3%) | 28,967 (19.3%) * |
| 0.90      | 0.60   | 2,700| 2,328  | 372    | 69,031 (46.0%) | 69,092 (46.0%) |
|           | 0.70   | 3,150| 2,778  | 372    | 55,531 (37.0%) | 55,592 (37.0%) |
|           | 0.75   | 3,375| 3,003  | 372    | 48,781 (32.5%) | 48,842 (32.5%) |
|           | 0.80   | 3,600| 3,228  | 372    | 42,031 (28.0%) | 42,092 (28.0%) |
|           | 0.85   | 3,825| 3,453  | 372    | 35,281 (23.5%) | 35,342 (23.6%) |
|           | 0.90   | 4,050| 3,678  | 372    | 28,531 (19.0%) | 28,592 (19.1%) |
|           | 0.95   | 4,275| 3,903  | 372    | 21,781 (14.5%) | 21,842 (14.6%) |
| 0.95      | 0.60   | 2,850| 2,463  | 387    | 64,531 (43.0%) | 64,592 (43.0%) |
|           | 0.70   | 3,325| 2,938  | 387    | 50,281 (33.5%) | 50,342 (33.5%) |
|           | 0.75   | 3,563| 3,176  | 387    | 43,156 (28.8%) | 43,217 (28.8%) |
|           | 0.80   | 3,800| 3,413  | 387    | 36,031 (24.0%) | 36,092 (24.1%) |
|           | 0.85   | 4,038| 3,651  | 387    | 28,906 (19.3%) | 28,967 (19.3%) |
|           | 0.90   | 4,275| 3,888  | 387    | 21,781 (14.5%) | 21,842 (14.6%) * |
|           | 0.95   | 4,513| 4,126  | 387    | 14,656 (9.8%)  | 14,717 (9.8%) * |

\[ \dagger \] The figures in parentheses show the ratio to the number of constraints and decision variables of the original problem.

We solve the problems with the combination of four kinds of \( \beta (= 0.80, 0.85, 0.90, 0.95) \) and seven kinds of \( k_1 (= 0.60, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95) \), and compare the optimal solutions. We set \( k_2 = 0.5 \). Table 2 shows the problem size. We solve the original problems, and we find the fact that the paths we should use to calculate the risk measure are included in the set \( G \)

\(^{13}\)In our examples, \( \rho^{(i)}_T = \rho^{(i)}_{1,T} \) because there is a risky asset, or \( n = 1 \).
if $|G|_1$ is larger than 2,622 for $\beta = 0.80$, 3,165 for $\beta = 0.85$, 3,499 for $\beta = 0.90$, and 3,694 for $\beta = 0.95$. We mark the rightmost column of the relevant cases with asterisks.

We show the results for $\beta = 0.8$ in Figure 18. The left-hand side shows the objective function values or the CVaR, and the right-hand side shows the computation time. The values for $k_1 = 0$ show the optimal solutions of the original problems. We can obtain the same optimal solutions as the ones of the original problems in the case where $k_1 = 0.6$ and $k_1 = 0.7$.

![Figure 18: CVaR and computation time($\beta = 0.8$)](image)

We cannot obtain the same optimal solutions as the original ones in the case where $k_1 = 0.75, 0.80, 0.85$, but the similar objective function values are derived. However, over-evaluated solutions are obtained in the cases for $k_1 = 0.90, 0.95$, and the objective function values get large sharply. The reason is that the paths we should remove from the set $G$ in order to calculate the CVaR are included in the set $G$ in the cases for $k_1 \leq 0.75$ as shown in Table 2. Computation time is reduced in proportion to the $k_1$ value. It takes about 370 seconds to solve the original problem. However, it takes about 75 seconds to solve the reduced problem for $k_1 = 0.7$, that is, the computation time is reduced to one-fifth.

![Figure 19: Error of CVaR and computation time ratio](image)
Figure 19 shows the results for four kinds of $\beta$ values in the case for $pf = 1$ and $np = 1$. The left-hand side shows the error ratios of the CVaR to the one of the original problems, and the right-hand side shows the ratio of the computation time to the one of the original problems. Even a small $k_1$ such as $k_1 = 0.6$, the error ratio of 0.055% is observed for $\beta = 0.95$. This is not the reason that the paths which should be removed from the set $G$ are included in the set $G$, but the reason that we replace Equation (26) for $i \in G$ with Equation (27). There are errors in the reduced problems because paths where the optimal cash $v_i^{(i)*}$ in the original problem is equal to the lower bound, or $v_i^{(i)*} = L_i$, are included in the set $G$. The larger the $\beta$ value is, the more largely the computation time is reduced. This is because the set $G$ associated with the larger $\beta$ includes more paths.

Figure 20: Optimal life insurance money

The left-hand side of Figure 20 shows the optimal life insurance money for $\beta = 0.8$, and the right-hand side of Figure 20 shows the error ratios for four kinds of $\beta$ values in the case for $pf = 1$ and $np = 1$. Large error ratios are observed for $k_1 = 0.90, 0.95$ as well as the objective function values, however they range between $\pm 1\%$.

Due to space limitation, we omit the investment units for a risky asset, but we show the error ratios of the investment units for four kinds of $\beta$ values in Figure 21. Large error ratios are observed for $k_1 = 0.90, 0.95$ as well as the others. Extremely large errors are observed for the combination of $\beta = 0.80$ and $k = 0.95$. About 10% errors are observed on average for $\beta = 0.95$.

We examine the error ratios of the optimal fire insurance money in Figure 22. About −40% of error is observed at time 27 for $\beta = 0.80$ and $k_1 \geq 0.75$, and about −100% of error is observed at time 29 for $\beta = 0.80$ and $k_1 \geq 0.85$. Even if the paths which should be used in calculating the risk measure are included in the set $G$, it does not affect the objective function value and the optimal life insurance money, however it affects the optimal fire insurance money drastically.

\[14\] If we do not change the associated constraints, we obtain the same optimal solutions as the original ones. However, the problem size gets large.
Figure 21: Error of investment units of a risky asset\((pf = 1, np = 1)\)

Figure 22: Error of fire insurance money\((pf = 1, np = 1)\)
7 Concluding remarks

In this paper, we discuss the optimization model for a household. We extend the studies in Hibiki, Komoribayashi and Toyoda (2005) for the practical use, and examine the model with numerical examples.

Term structures of income and expenditure of the household are affected by the householder’s death. We propose a model involving three factors we need to consider if the householder is dead to analyze it with practical cash flow streams. Three factors are receipt of survivor’s pension, exemption from the home loan payments, and change of the consumption level. If the household receives the survivor’s pension, the lower life insurance money can be bought. However, exemption from mortgage payment does not affect the objective function value and life insurance money. The reason is that the loan payment is not forgiven if the house is bought after the householder is dead. Exemption from mortgage payment or the reduction in loan payment contributes to the increase in the terminal financial wealth. When the household keeps the consumption level lower, the CVaR gets higher and the household purchases the lower life insurance money. This is because the household can hedge risk against the loss of wage income by the life insurance, and the reduction in consumption can make up for the loss of wage income caused by the householder’s death. Three factors considered in this paper affect the financial wealth, and we find these are important factors to construct the model.

It is a big event in the life for the household to purchase a house. We analyze the sensitivity of four kinds of parameters associated with home buying; time, down payment, loan period, and mortgage interest rate. We clarify that parameters also affect the optimal solutions.

In order to solve problems fast, we show the model with the grouped path. We propose the method to determine the set, and we test the method with numerical examples. We have almost the same solutions as the original ones, but the computation time is reduced to one-fifth on average, and one-tenth at a maximum.

We use the practical examples for a household, and we obtain the results which coincide our feeling. The extended model derives the optimal insurance and investment strategies appropriately, and we can use the model for giving a financial advice to individual investors.

References


( The paper written in English can be downloaded from http://www.ae.keio.ac.jp/lab/soc/hibiki/profile/Hibiki_IFORS2005.pdf )


