Optimal Symmetric No-trade Ranges in Asset Rebalancing Strategy with Transaction Costs

— An application to the Government Pension Investment Fund in Japan —

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Abstract There are many studies of optimal asset allocation with transaction costs in academic literatures. However, those are numerically solved for two-asset and three-asset cases. In contrast, the investment in five asset classes (domestic bond and stock, international bond and stock, and cash) is at least required for practical pension fund management. Therefore, there are some real limitations to the continuous-time approach used in the previous literatures, which are the methods of solving the HJB equation for the stochastic control problem. In general, most investors use the rebalance rule with the no-trade ranges in practice which are constant and symmetric to the policy asset mix because it is easy to use the rule. In this paper, we propose the optimization model for the multiple asset allocation problem with transaction costs to determine the symmetric no-trade ranges of the policy asset mix using the DFO approach proposed by Hibiki et al. (2013). Specifically, we solve the five-asset problem with boundary constraints for cash for the GPIF (Government Pension Investment Fund) in a discrete-time and finite-period setting. We clarify the fact that we need to adjust the amounts of risky assets even within the no-trade range if the boundary constraints for cash are required, and we describe the simulation procedure in the discrete-time model. We examine the difference for various time intervals and horizons, and conduct the sensitivity analysis for the various proportional transaction cost rates, the tracking error aversions, and the bounds for cash constraints. In addition, we compare the optimal time-dependent no-trade ranges with the constant no-trade ranges. The numerical results show the possibilities of applying the DFO model to the practical problem determining the symmetric no-trade ranges.

Keywords: asset allocation, no-trade range, derivative free optimization

1. Introduction

The Government Pension Investment Fund (GPIF) needs to manage its fund based on the target allocation of policy asset mix, which is computed using mean values, standard deviations of each asset and correlations of the rate of return among assets. Almost corporate pension funds also manage their funds in the same way, though their policy asset mixes are different each other depending on the degree of maturing, guaranteed interest rate, and so on.

The actual weights of assets diverge from the target weights in the portfolio as the asset prices change randomly. Meanwhile, the transaction costs need to be paid to maintain the target weights by rebalancing assets. Therefore the investors do not rebalance their assets without exceeding the boundary of the no-trade region because it exists the trade-off between the costs associated with tracking error (deviations from the target allocation) and the transaction costs. As an example, the target allocation of policy asset mix and permissible ranges of deviation from the target of the GPIF are shown in Table 1.

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Table 1: Policy asset mix of the GPIF

<table>
<thead>
<tr>
<th>Policy asset mix</th>
<th>Domestic bonds</th>
<th>Domestic stocks</th>
<th>International bonds</th>
<th>International stocks</th>
<th>Short-term assets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Target allocation</td>
<td>67%</td>
<td>11%</td>
<td>8%</td>
<td>9%</td>
<td>5%</td>
</tr>
<tr>
<td>Permissible range</td>
<td>±8%</td>
<td>±6%</td>
<td>±5%</td>
<td>±5%</td>
<td>—</td>
</tr>
</tbody>
</table>

The asset weights of the target portfolio in almost all pension funds are usually calculated by using the mean-variance model as well as the GPIF. However, it is unknown for the pension funds how to decide the no-trade ranges of their portfolios.

There are many academic studies for determining the boundaries of no-trade region and the optimal asset rebalancing strategy with transaction costs. Leland[10] formulates the stochastic control problem in a continuous-time setting, and derives the optimal rebalancing strategy for an investor who minimizes both proportional transaction costs and tracking error with respect to a specified target asset mix for the three-asset case. Donohue and Yip[5] conduct the sensitivity analysis of the modeling parameters associated with Leland[10], and compare the different rebalancing strategies. Pliska and Suzuki[14] improve Leland model[10], and derive the optimal trading strategy for an investor who minimizes both the fixed and proportional transaction costs and the tracking errors. Impulse control theory is used in a continuous-time, dynamic setting to deal with the problem in a general and analytical way, and the explicit solution for the two-asset case is derived.

Moreover, there are some continuous-time and discrete-time models in finite cases with the maximization of the expected CRRA utility of wealth. In a continuous-time approach, Liu and Loewenstein[11] examine the optimal trading strategy for a CRRA investor who trades two assets (bond and stock), and maximizes the expected utility of wealth on a finite date and with transaction costs. The closed-form solutions are obtained at an uncertain time by solving the Hamilton-Bellman-Jacobi (HJB) equation. Lynch and Tan[12] numerically solve the decision problem of a multi-period CRRA investor who faces fixed and proportional transaction costs and has access to two risky assets, both with predictable returns. Atkinson and Ingpochai[1] examine the intertemporal optimal portfolio selection and the consumption rule of a CRRA investor who faces the proportional transaction costs when trading between a risk-free asset and N risky assets. Numerical examples for a portfolio with a risk-free asset and two risky assets are provided for constant variance as well as stochastic variance. Using a discrete-time approach, Gennotte and Jung[6] solve the problem with a risky asset and a riskless asset. They derive the optimal trading strategies described in terms of no transaction region using the binomial model. Boyle and Lin[2] extends the work by Gennotte and Jung[6], and derive an explicit closed-form solution for the investor who has a power utility function.

Some different types of the problems are formulated for N-asset cases, but only two-asset or three-asset problems are solved numerically. However, almost all investors of pension funds and investment trusts invest in more than five assets, such as domestic bond and stock, international bond and stock, and short-term asset (cash). There are some real limitations to solve the problem for the five-asset case in a continuous-time model by the HJB equation. In a real world, it is difficult to employ the utility functions assumed in the previous studies of finite horizon models. In addition, most institutional investors determine and manage the policy asset mix. We need to formulate a discrete-time model because we cannot trade assets continuously. In this paper, we formulate the finite and discrete-time model...
in the Leland approach which involves the transaction costs and the tracking errors in the objective function as well as Hibiki et al.[8]. We solve the problem with DFO(derivative free optimization), which is one of the mathematical programming approach. We can describe the simulation procedure flexibly in the DFO approach.

Hibiki et al.[8] derive the no-trade region with the DFO approach for the Leland model[10] in a discrete time and finite horizon setting, and show the usefulness of this approach. The optimal no-trade boundaries in a finite case are time-dependent. On the other hand, most investors use the rebalance rule with the no-trade ranges in practice which are constant and symmetric to the policy asset mix because it is easy to use the rule. In this paper, we propose the approach to derive the constant and symmetric no-trade ranges of the target portfolio of multiple assets numerically as with the no-trade ranges of the GPIF. In addition, we propose the model with the time-dependent no-trade ranges, and thus we compare them with the constant ranges.

We have two original contributions in this paper as follows.

(1) Deriving the symmetric no-trade ranges, specifically for the GPIF parameters

There are no models to derive the symmetric no-trade ranges with respect to the policy asset mix for multiple-asset case used in practice because the practical use is not considered in the previous literatures. In this paper, we solve the five-asset problem with the GPIF parameters, and derive the optimal solutions.

(2) Adjustment algorithm for the rebalancing strategy

We clarify the fact that we need to adjust the amounts of risky assets even within the no-trade ranges if the boundary constraints for cash are required. We describe the simulation procedure involving the adjustment algorithm in the discrete-time model.

This paper is organized as follows. In Section 2, we clarify the fact that we need to adjust the amounts of risky assets even within the no-trade ranges, and we describe the simulation procedure in the discrete-time model. We solve the optimization problem to decide the no-trade(permissible) ranges with respect to the policy asset mix using the parameters applied for the GPIF, and examine them in Section 3. In Section 4, we derive the optimal time-dependent no-trade ranges, and compare them with the constant ranges. Section 5 provides our concluding remarks.

2. Determination of the no-trade ranges and DFO modeling

In this paper, we solve the problem to decide the no-trade ranges with the DFO method as well as Hibiki et al.[8]. The DFO method is the non-linear optimization method where the problems are solved without the derivative of the objective function\(^1\). The optimal solution can be obtained precisely for the problem where the minimized objective function is convex, and the number of decision variables is several. The number of decision variables is \(2^N N\) in the \(N\)-asset case of the Leland model[10], and increases exponentially with \(N\). However, the no-trade ranges of the GPIF in Table 1 are symmetric to the target allocation, and therefore the number of decision variables is \(N\) which is the same as the number of risky assets. The type of the problem goes well with the DFO method.

We describe the simulation procedure of calculating the objective function value to derive

\(^1\)The readers can refer Conn, Scheinberg and Vicente[3] and the chapter nine of Nocedal and Wright[13] in detail. In this paper, we use NUOPT/DFO[16] added on the mathematical programming software package called NUOPT developed by Mathematical System, Inc. NUOPT/DFO implements the trust-region methods based on derivative free models, which maintain quadratic models based only on the objective function values computed at sample points.
the optimal solution by the DFO method. We discretize the stochastic process for the risky assets used in Leland[10], and generate the normal random number for \( \varepsilon_i \) by using the Monte Carlo method.

\[
\frac{\Delta S_i}{S_i} = \mu_i \Delta t + \sigma_i \sqrt{\Delta t} \varepsilon_i, \quad \varepsilon_i \sim N(0,1),
\]

where \( S_i \) is the price of asset \( i \), \( \mu_i \) is the expected rate of return of asset \( i \), \( \sigma_i \) is the standard deviation of asset \( i \), and \( \rho_{ij} \) is the correlation coefficient between asset \( i \) and \( j \). We calculate the weight of risky assets from the price, and rebalance the assets based on the rule. We formulate the model to derive the no-trade ranges with respect to the given policy asset mix by minimizing the sum of the transaction costs and the tracking errors. We do not involve the no-trade range of asset \( N \), because we impose the boundary constraints for cash where the lower bound of the asset \( N \) is \( L_N \), and the upper bound is \( U_N \).

2.1. Imposing the boundary constraints for cash

When we rebalance a risky asset out of the no-trade range to the boundary, there are some cases that we need to adjust the amount of other risky assets even within the no-trade ranges\(^3\). An example is shown in Table 2. Suppose we have the target portfolio at time 0, and set up \( L_N = 3\% \). Assume that the time passed and asset prices changed. At time \( t \), we assume that the weight of the domestic bond becomes 53\%, and the weight of the domestic stock becomes 19\%. Because both assets are beyond the no-trade ranges, the weight of the domestic bond is moved from 53\% to 59\% which is the lower no-trade boundary, and the weight of the domestic stock is moved from 19\% to 17\% which is the upper no-trade boundary. In contrast, we do not need to rebalance the international bond and stock if we follow the rebalance rule because both assets are within the no-trade ranges. As a result, the calculated weight of cash is \(-3\%\), but we need to have 3\% cash. Therefore, we have to decrease 6\% of the risky assets at least to increase 6\% cash. This occurs because the asset weight substantially below the lower no-trade boundary needs to be increased. If we do not prevent this situation by decreasing other assets which are over the target weight, the total weight of the assets except asset \( N \) becomes beyond \( 1 - L_N \). In the case of Table 2, all of risky assets can be within the no-trade ranges by decreasing 2\% of the assets even within the no-trade ranges respectively. This is implemented by the allocation moves from "Rebalance (1)" to "Rebalance (2)". We involve the adjustment algorithm with the above-mentioned steps to decide the no-trade ranges as follows\(^4\).

1. We calculate the excess weight \( EW \) which needs to be adjusted.
2. We examine the set \( I \) of assets above the target weight of policy asset mix, and calculate \( EX_i \) which denotes the difference between the investment weight of asset \( i \) and the target weight. One asset is at least above the target weight.

\(^2\)In the case of the GPIF as in Table 1, the asset \( N \) is short-term asset or cash. We need to have cash in a degree in order to pay for pension money, while we need to invest in risky assets without having much cash because of the efficient investment. When we do not impose the boundary constraints, we have \( L_N = 0 \) and \( U_N = 1 \).

\(^3\)We do not need to adjust the risky assets within the no-trade ranges in the continuous-time model because even the asset out of the no-trade range is rebalanced on the boundary immediately. However, we need to adjust the risky assets within the no-trade ranges even in the continuous-time model if the lower bound of asset \( N \) is greater than zero, that is \( L_N > 0 \).

\(^4\)If the no-trade ranges are small, we do not need to adjust the amounts of assets, however we need to pay much transaction costs. Therefore, the optimal no-trade ranges are derived in consideration of the trade-off.
Table 2: Example of adjustment steps within the no-trade region

<table>
<thead>
<tr>
<th>Asset class</th>
<th>Domestic bond</th>
<th>Domestic stock</th>
<th>International bond</th>
<th>International stock</th>
<th>Short-term asset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range</td>
<td>59~75%</td>
<td>5~17%</td>
<td>3~13%</td>
<td>4~14%</td>
<td>—</td>
</tr>
<tr>
<td>time 0</td>
<td>67%</td>
<td>11%</td>
<td>8%</td>
<td>9%</td>
<td>5%</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>time t</td>
<td>53%</td>
<td>19%</td>
<td>13%</td>
<td>14%</td>
<td>1%</td>
</tr>
<tr>
<td>Rebalance (1)</td>
<td>59%</td>
<td>17%</td>
<td>13%</td>
<td>14%</td>
<td>-3%</td>
</tr>
<tr>
<td>Rebalance (2)</td>
<td>59%</td>
<td>15%</td>
<td>11%</td>
<td>12%</td>
<td>3%</td>
</tr>
</tbody>
</table>

(3) We solve the problem to minimize the sum of transaction costs and tracking errors at each time and on each path in order to obtain the optimal decreasing weight of asset $i$ or $\Delta w_i$.

$$\sum_{i \in I} \Delta w_i = EW, \quad 0 \leq \Delta w_i \leq EX_i$$

When the cash is beyond the upper limit, the amounts of risky assets are adjusted based on the similar ways. We omit the algorithm because of the similar steps.

2.2. Simulation procedure

Let $q_i (i = 1, \ldots, N - 1)$ denote the no-trade range of asset $i$. We describe the simulation procedure and solve the optimization problem by the DFO method in what follows. The superscript of 'b' shows the notation of pre-rebalance, and the superscript of 'a' shows the notation of post-rebalance. We let $p_i$ the target weight of asset $i$ of the policy asset mix.

(1) Suppose that the amount of wealth at time 0 is 1 or $W^{(m)}_0 = 1$, and the value of asset $i$ is $p_i$ or $S^{(0)}_{i,0} = p_i$.

(2) We calculate sequentially the value of asset $i(S_{i,t}^{(m)})$, the weight($w_{i,t}^{(m)}$) and the amount of wealth($W^{(m)}_t$) on path $m$ at time $t$ ($t = 1, \ldots, T; m = 1, \ldots, M; i = 1, \ldots, N$). We generate random samples $\varepsilon^{(m)}_{i,t}$ in Equation (3) by the Monte Carlo method. $S^{(m)}_{i,t-1}$ is computed in Equation (19) to be described later.

$$S^{b(m)}_{i,t} = \left(1 + \mu_i \Delta t + \sigma_i \sqrt{\Delta t} \varepsilon^{(m)}_{i,t} \right) S^{a(m)}_{i,t-1} \quad (3)$$

$$W^{(m)}_t = \sum_{i=1}^{N} S^{b(m)}_{i,t} \quad (4)$$

$$w_{i,t}^{b(m)} = \frac{S^{b(m)}_{i,t}}{W^{(m)}_t} \quad (5)$$

We compute the weight of asset $i$ after rebalancing ($w_{i,t}^{a(m)}$) as in Equations (6) and (7) based on the value of $w_{i,t}^{b(m)}$,

$$w_{i,t}^{a(m)} = \begin{cases} p_i - q_i & \text{for } w_{i,t}^{b(m)} \in [0, p_i - q_i), \\ w_{i,t}^{b(m)} & \text{for } w_{i,t}^{b(m)} \in [p_i - q_i, p_i + q_i], \quad (i \in I_{N-1}) \\ p_i + q_i & \text{for } w_{i,t}^{b(m)} \in [p_i + q_i, 1], \end{cases} \quad (6)$$

$$w_{N,t}^{a(m)} = 1 - \sum_{i \in I_{N-1}} w_{i,t}^{a(m)} \quad (7)$$

The investment value is not dependent on path $m$ at time 0, but we describe the notation for convenience.
where $I_{N-1} = \{1, 2, \ldots, N-1\}$ denote the set of assets except asset $N$. As discussed in Section 2.1, if the weight of asset $N$ ($w_{Nt}^{(m)}$) is beyond the upper or lower limit when rebalancing the assets except asset $N$ on the boundary of the no-trade range, we need to adjust the weights of the assets except asset $N$ in what follows.

1. The case that the weight of asset $N$ is below the lower limit, or $w_{Nt}^{a(m)} < L_N$.

We calculate the excess weight or $EW_t^{(m)} = L_N - w_{Nt}^{a(m)}$, and adjust the asset weights as follows.

a. Let $I_t^{+(m)}$ denote the set of assets which meets the condition that $w_{it}^{a(m)} > p_i$ ($i \in I_{N-1}$).

b. We solve the following subproblem of minimizing the objective function to obtain the values of $\Delta w_{it}^{(m)}$ which are the amounts of the decreases in the asset weights in the set $I_t^{+(m)}$.

\[
\text{Minimize} \quad C(\Delta w_{it}^{(m)}) - C(0), \quad \text{(8)} \\
\text{subject to} \quad 0 \leq \Delta w_{it}^{(m)} \leq EX_{it}^{(m)} (i \in I_t^{+(m)}), \quad \text{(9)} \\
\sum_{i \in I_t^{+(m)}} \Delta w_{it}^{(m)} = EW_t^{(m)}, \quad \text{(10)} \\
\Delta w_{it}^{(m)} = 0 (i \in I_{N-1} - I_t^{+(m)}), \quad \text{(11)} \\
\Delta w_{tN}^{(m)} = -EW_t^{(m)}, \quad \text{(12)}
\]

where $\Delta w_{it}^{(m)} = (\Delta w_{1t}^{(m)}, \ldots, \Delta w_{Nt}^{(m)})$, $EX_{it}^{(m)}$ is the investment weight minus the target weight or $EX_{it}^{(m)} = w_{it}^{a(m)} - p_i$, and $0$ is the $N$-dimensional zero vector. According to the after-mentioned Equation (20), $C(\mathbf{x})$ is the objective function on path $m$ at time $t$. We can write it as follows,

\[
C(\mathbf{x}) = \lambda \sum_{i=1}^{N} \sum_{j=1}^{N} (EX_{it}^{(m)} - x_i) (EX_{jt}^{(m)} - x_j) \sigma_{ij} + \sum_{i \in I_{N-1}} k_i x_i. \quad \text{(13)}
\]

c. Let $\Delta w_{it}^{+(m)}$ denote the optimal solution of the subproblem, and we replace $w_{it}^{a(m)} - \Delta w_{it}^{+(m)}$ with $w_{it}^{a(m)}$ adjusted to avoid using the another notations.

2. The case that the weight of asset $N$ is over the upper limit, or $w_{Nt}^{a(m)} > U_N$.

We calculate the deficit weight or $DW_t^{(m)} = w_{Nt}^{a(m)} - U_N$, and adjust the asset weights as follows.

a. Let $I_t^{-(m)}$ denote the set of assets which meets the condition that $w_{it}^{a(m)} < p_i$ ($i \in I_{N-1}$).

b. We solve the following subproblem of minimizing the objective function to obtain the values of $\Delta w_{it}^{-(m)}$ which are the amounts of the increases in the asset weights in the set $I_t^{-(m)}$.

\[
\text{Minimize} \quad C(\Delta w_{it}^{-(m)}) - C(0), \quad \text{(14)} \\
\text{subject to} \quad 0 \leq \Delta w_{it}^{-(m)} \leq DE_{it}^{(m)} (i \in I_t^{-(m)}), \quad \text{(15)}
\]

As in Equation (11), the decreases in the asset weights except the assets in the set $I_t^{+(m)}$ and asset $N$ are equal to zero, or $\Delta w_{it}^{-(m)} = 0$. 

\[6\text{As in Equation (11), the decreases in the asset weights except the assets in the set } I_t^{+(m)} \text{ and asset } N \text{ are equal to zero, or } \Delta w_{it}^{-(m)} = 0.\]
\[
\sum_{i \in I_t^{-}(m)} \Delta w^{(m)}_{it} = DW_t^{(m)},
\]
\[
\Delta w^{(m)}_{it} = 0 \ (i \in I_{N-1} - I_t^{-}(m)),
\]
\[
\Delta w^{(m)}_{Nt} = -DW_t^{(m)},
\]

where \(DF_t^{(m)}\) is the target weight minus the investment weight or \(DF_t^{(m)} = p_t - w^{a(m)}_{it}\).

c. We replace \(w^{a(m)}_{it} + \Delta w^{(m)}_{jt}\) with \(w^{a(m)}_{it}\) adjusted.

3. The value of asset \(i\) after rebalancing \((S_{it}^{a(m)})\) is calculated in Equation (19),
\[
S_{it}^{a(m)} = w_{it}^{a(m)} W_t^{(m)} \ (i = 1, \ldots, N).
\]

3. Application to the GPIF

The GPIF manages asset allocation on the basis of the policy asset mix for five assets as in Table 1: domestic bond(DB), domestic stock(DS), international bond(IB), international stock(IS), and short-term asset(SA). Different three cases of the appreciation rate for total factor productivity are shown in Ministry of Health, Labour and Welfare[17]. In this paper, we use the expected rates of return estimated for the intermediate case. We examine the model using the expected values, the standard deviation and the correlation coefficients of the rates of return shown in Table 3. The other parameters are as follows.

\[
\begin{align*}
\sum_{i \in I_t^{-}(m)} \Delta w^{(m)}_{it} &= DW_t^{(m)}, \\
\Delta w^{(m)}_{it} &= 0 \ (i \in I_{N-1} - I_t^{-}(m)), \\
\Delta w^{(m)}_{Nt} &= -DW_t^{(m)},
\end{align*}
\]

The GPIF manages asset allocation on the basis of the policy asset mix for five assets as in Table 1: domestic bond(DB), domestic stock(DS), international bond(IB), international stock(IS), and short-term asset(SA). Different three cases of the appreciation rate for total factor productivity are shown in Ministry of Health, Labour and Welfare[17]. In this paper, we use the expected rates of return estimated for the intermediate case. We examine the model using the expected values, the standard deviation and the correlation coefficients of the rates of return shown in Table 3. The other parameters are as follows.

\[
\begin{align*}
\sum_{i \in I_t^{-}(m)} \Delta w^{(m)}_{it} &\quad DW_t^{(m)}, \\
\Delta w^{(m)}_{it} &\quad 0 \ (i \in I_{N-1} - I_t^{-}(m)), \\
\Delta w^{(m)}_{Nt} &\quad -DW_t^{(m)},
\end{align*}
\]
- Proportional transaction cost rates
  (DB) $k_1 = 0.2\%$, (DS) $k_2 = 1.2\%$, (IB) $k_3 = 0.2\%$, (IS) $k_4 = 1.2\%$
- Tracking error aversion : $\lambda = 3$
- Boundary constraints for short-term asset (cash constraints)
  (lower bound) $L_N = 3\%$, (upper bound) $U_N = 10\%$
- Number of sample paths : $M = 10,000$

<table>
<thead>
<tr>
<th></th>
<th>Domestic bond(DB)</th>
<th>Domestic stock(DS)</th>
<th>International bond(IB)</th>
<th>International stock(IS)</th>
<th>Short-term asset(SA)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected return</td>
<td>3.7%</td>
<td>6.0%</td>
<td>3.7%</td>
<td>6.2%</td>
<td>2.2%</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>5.45%</td>
<td>22.25%</td>
<td>13.44%</td>
<td>19.85%</td>
<td>3.71%</td>
</tr>
<tr>
<td>Correlation</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Domestic bond</td>
<td>1.00</td>
<td>0.15</td>
<td>-0.06</td>
<td>-0.05</td>
<td>0.45</td>
</tr>
<tr>
<td>Domestic stock</td>
<td>0.15</td>
<td>1.00</td>
<td>-0.26</td>
<td>0.27</td>
<td>-0.01</td>
</tr>
<tr>
<td>International bond</td>
<td>-0.06</td>
<td>-0.26</td>
<td>1.00</td>
<td>0.55</td>
<td>-0.05</td>
</tr>
<tr>
<td>International stock</td>
<td>-0.05</td>
<td>0.27</td>
<td>0.55</td>
<td>1.00</td>
<td>-0.12</td>
</tr>
<tr>
<td>Short-term asset</td>
<td>0.45</td>
<td>-0.01</td>
<td>-0.05</td>
<td>-0.12</td>
<td>1.00</td>
</tr>
</tbody>
</table>

### 3.1. Numerical analysis

We examine the model for 28 combinations of four kinds of time intervals ($\Delta t = 0.1, 0.2, 0.5, 1.0$) and seven kinds of time horizons (5, 10, 15, 20, 30, 40, 50 years)\(^\text{10}\). Figure 1 is shown to compare the results for different time horizons, and Figure 2 is shown to compare the results for different time intervals.

Figure 1 shows that the no-trade ranges converge to certain values as the time horizon is larger. The shorter the time horizon is, the larger the no-trade ranges are on average because investors tend to adopt the no-trade strategy to cut the trading costs on the rebalancing point close to the time horizon. Figure 2 shows that the no-trade ranges are reduced as the time interval is large. The investors reduce the no-trade range and avoid the increase in the tracking error calculated by the quadratic function because the investors cannot trade assets except the time points in the discrete-time model. These results can be also shown in the case of more than three assets as well as in Hibiki \textit{et al.}\cite{8}. The relationship among the no-trade ranges of four risky assets is

$$DB > DS > IS > IB.$$  

It is the same as the no-trade permissible ranges of the GPIF.

\(^{10}\)The number of periods is calculated by years divided by the time intervals. For example, the number of periods is 100 when $\Delta t = 0.1$ and 10 years.
Figure 1: Comparison of the results for different time horizons

Figure 2: Comparison of the results for different time intervals
This shows that the policy asset mix affects their no-trade ranges\(^\text{11}\). We calculate the different no-trade ranges for their target portfolios and show Figure 3 to verify the reasonableness of the results. We derive ten kinds of target portfolios for their expected rates of return using the mean-variance model with parameters in Table 3\(^\text{12}\). We show the no-trade ranges corresponding to the portfolios in the left-hand side of Figure 3, and the relationship between the portfolio weights and the no-trade ranges in the right-hand side of Figure 3. The expected rate of return of P1 is the lowest, and that of P10 is the highest in order. As the results, the weights of DB become small and the weights of DS and IS have large and similar values in proportion as the expected rates of return of the portfolios become high. The weights of IS becomes small toward the P10 after achieving the peak for P3.

![No-trade ranges of different portfolios](attachment:image1)

![The relationship between weights and no-trade ranges](attachment:image2)

**Figure 3: No-trade ranges for different target portfolios**

The right-hand side of Figure 3 shows that the no-trade ranges increase as the weights approach 50%, and the no-trade ranges decrease as the weights lose touch with 50%. In particular, the no-trade ranges of DS are similar to those of IS, and they become large as the expected rates of return of the portfolios increase because their weights are less than 50%. The no-trade ranges of DB are getting large for P1 to P7 portfolios, but they become small after achieving the peak at P7. The no-trade ranges of IB become small from P3 portfolio. Meanwhile, the no-trade ranges of IB are getting large for P1 to P3 portfolios, but they become small after achieving the peak at P3.

We examine the sensitivity of the optimal no-trade range of each asset to the objective function value. In particular, we calculate the corresponding objective function to the

\(^{11}\)Figure 7 shows that the no-trade ranges become wide as the proportional transaction cost rates increase. The no-trade ranges of the stocks become wide relatively to the bonds because the proportional transaction cost rates of the stocks are 1.2%, and those of the bonds are 0.2%. The no-trade range is more affected by the target ratio than the proportional transaction cost for the domestic bond. We conduct the sensitivity analysis for the proportional transaction cost in detail in Section 3.2.

\(^{12}\)The GPIF uses the following constraints to derive the target portfolio; the target ratio of short-term asset (cash) = 5%, the expected rate of return of the target portfolio = 4.1%, and the relationship among the target weights of the assets is DS > IS > IB. In particular, we impose constraints on the above-mentioned relationship among asset weights as follows; DS weight ≥ IS weight + 1% and IS weight ≥ IB weight + 1%. We solve the problem by the mean-variance model, and obtain the optimal weights of 66.7% for DB, 10.4% for DS, 8.4% for IB and 9.4% for IS. When we round the weights to 1% unit, the weights of the target portfolio in Table 1 are obtained. The policy portfolio is the P3 portfolio. The other portfolios (P1 to P10 except P3) can be also derived by solving the mean-variance problems with the expected rates of return of 3.9%, 4.0%, 4.3%, 4.5%, 4.7%, 4.9%, 5.1%, 5.3% and 5.5%, respectively.
different no-trade ranges of the certain asset except which we fix the optimal no-trade ranges of other assets. We show the graphs in Figure 4 where the horizontal axis is the multiple of the optimal no-trade range, and the vertical axis is the multiple of the objective function value. The values of 1 on the both axes correspond to the optimal values. Figure 4 shows the results for the case of $\Delta t = 0.1$ and 5 years on the left-hand side, and for the case of $\Delta t = 0.1$ and 10 years on the right-hand side.

Figure 4: Objective function values for different no-trade ranges

The degrees of convexity of the objective function are DS, IS, IB, and DB in order. This order is the same as that of their volatilities. The increase in tracking error needs to be larger than the decrease in the transaction cost to obtain the convex objective function without degeneracy because the transaction costs approach zero as the no-trade ranges become large. The volatilities of the assets affect the tracking error, and therefore the order of the degrees of convexity is the same as that of the volatilities. As the no-trade ranges become large, the objective function is not degenerate, but the increase in the objective function values for IB and DB get smaller. This means that the sensitivity of the no-trade ranges to the objective function is small and we have the possibilities of solving the problems unstably. We need to pay attention to it when solving the problems.

Figure 5 shows the objective function values for the different no-trade ranges of DB and IB calculated under four kinds of time intervals with 10 years, and four kinds of years with $\Delta t = 0.1$. The sensitivity of the objective functions for DB around the optimal solution are similar even if the time intervals or the number of years are different. But the objective function value converges to the certain value if the no-trade range is larger than the certain ranges(certain multiple of the horizontal axis in Figure 5). As the number of years and the time intervals are smaller, the objective function value converges to the certain value at the smaller no-trade range. The objective function value is insensitive for the larger no-trade range of DB. The sensitivity of the objective function value becomes small because the policy ratio of IB is 8% which is smaller than the other policy ratios.
We examine the probability that the rebalance is needed to impose the cash constraints even if the asset weights are within the no-trade ranges. We call it cash adjusted probability. We show the probability and the computation time in Figure 6. The cash adjusted probability is defined as the sum of the probabilities that the weight of asset $N$ is below the lower bound $L_N$ and above the upper bound $U_N$. It is calculated by Equation (23) using the excess weight $EW_t^{(m)}$ and the deficit weight $DW_t^{(m)}$ in the simulation procedure of section 2.2. We denote $1_A$ the indicator function which value is 1 if the condition $A$ is satisfied, or 0 otherwise.

$$\varphi = \frac{1}{T \cdot M} \sum_{t=1}^{T} \sum_{m=1}^{M} \left( 1_{\{EW_t^{(m)} < 0\}} + 1_{\{DW_t^{(m)} > 0\}} \right)$$

(23)

The cash adjusted probability becomes small for the smaller time interval or the longer time horizon because the variability of the risky asset for a period is relatively small. The computation time increases for the small time interval and the longer time horizon because the number of periods increases.
3.2. Sensitivity analysis

We conduct the sensitivity analysis for three kinds of parameters (proportional transaction cost rate of stocks $k_2$ and $k_4$, tracking error aversion $\lambda$ and cash constraint parameter $\alpha$) as follows.

\[
k_2 = k_4 = 0.2\%, 0.6\%, 1.0\%, 1.2\%, 1.5\%, 2.0\%
\]

\[
\lambda = 1, 2, 3, 4, 5, 10, 20, 30
\]

\[
\alpha = 0.0, 0.5, 1.0, 1.5, 2.0, 2.5
\]

We set the lower bound $L_N$ and the upper bound $U_N$ using the following equations,

\[
L_N = (1 - \alpha)p_N + \alpha L_0^N \quad \text{for lower bound},
\]

\[
U_N = (1 - \alpha)p_N + \alpha U_0^N \quad \text{for upper bound},
\]

where $L_0^N = 3\%$, $U_0^N = 10\%$, $p_N = 5\%$, $\Delta t = 0.1(\text{year})$ and the time horizon is 10 years.

We show the results of the sensitivity analysis in Figures 7 to 9. The left sides include the no-trade ranges of the four risky assets and the right sides include the objective function values, transactions costs and tracking errors.

![Fig. 6: Cash adjusted probabilities and computation time](image)

Figure 6: Cash adjusted probabilities and computation time

![Fig. 7: Sensitivity analysis(1)](image)

Figure 7: Sensitivity analysis(1) proportional transaction cost rate
Figure 7 shows that the transaction costs increase in proportion to the cost rates. However, the no-trade ranges become large to avoid the trading assets, and tracking error increases simultaneously. Figure 8 shows that the no-trade ranges decrease to keep the tracking error small as the tracking error aversion increases. The no-trade ranges are small but consequently the transaction costs increase because of the frequent trading. Figure 9 shows that the no-trade ranges of DB become large, but those of other assets become small as the bound of the cash constraints (or the multiple $\alpha$) becomes large. When the multiple $\alpha$ is more than one which shows the lower bound is 3% and the upper bound is 10%, the no-trade ranges become insensitive to the bounds of constraints.

### 3.3. The comparison of the optimal no-trade ranges with the actual weights

We examine the actual weights published quarterly from March 2008 to December 2012 by the GPIF[7]. Figure 10 shows the actual weights (‘Actual W’), the policy asset mix (‘PAM’) and the no-trade ranges of the GPIF (‘GPIF’), and the optimal no-trade ranges for 5 and 50 years solved by the DFO method using the parameters of the basic analysis in Section 3.1, where the rebalancing interval is a month ($\Delta t = \frac{1}{12}$).

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13The GPIF has managed their investment assets using the no-trade ranges in Table 1 since March 2008.
Figure 10: The optimal no-trade ranges and the actual weights

The actual weights have moved within the no-trade ranges set by the GPIF because their ranges are relatively large. The optimal no-trade ranges derived in this paper are smaller than those of the GPIF. The actual weights have moved almost just within the optimal no-trade ranges except four cases\(^{14}\), and thus the reasonable ranges can be derived to some extent\(^{15}\).

4. Time-dependent no-trade ranges in a finite horizon

4.1. Finite horizon model

In this paper, we formulated the model in Section 2 and derived the optimal constant and symmetric no-trade ranges in the numerical analysis in Section 3 because it is easy to employ the rebalance rule in practice with the constant and symmetric no-trade ranges with respect to the policy asset mix. The results in subsection 3.3 show the optimal no-trade ranges are reasonable in examining the actual weights. Meanwhile, the previous studies show that the optimal no-trade boundaries are time-dependent in a finite horizon. Then we derive the optimal time-dependent and symmetric no-trade ranges, and compare them with the constant and symmetric ranges.

Hibiki et al.[8] formulate the model with the time-dependent no-trade boundaries in the Leland approach[10] which minimizes the objective function involving the transaction costs and the tracking errors. The boundaries are expressed by the exponential functions to show

\(^{14}\)Four cases are IS on December 2012 for 5 year problem, IS on September and December 2012 for 50 year problem, and DB and IB on December 2012 for 50 year problem. The probability of the violation is 1.25\%\((=1/80)\) for 5 year problem, and 3.75\%\((=3/80)\) for 50 year problem.

\(^{15}\)It is difficult to evaluate the validity of the optimal ranges because the optimal values depend on the parameters such as the proportional transaction costs and the tracking error aversions. This is one of the ways to evaluate the validity.
that the ranges between two boundaries are large as the time to maturity (time horizon) becomes small. It is consistent with the previous studies for the problems with the CRRA utility function [6, 11]. In this paper, we express the time-dependent no-trade boundaries \( q_i(t) \) in Equation (24) by reference to Hibiki et al. [8],

\[
q_i(t) = \alpha_i \left\{ 1 + \gamma e^{-\beta(T-t+\Delta t)} \right\}, \quad (i = 1, \ldots, N - 1),
\]

where \( \alpha_i, \beta \) and \( \gamma \) are the parameters determining the ranges. When \( T \to \infty \),

\[
\lim_{T \to \infty} q_i(t) = \alpha_i,
\]

and the equations can express the constant boundaries in the infinite horizon. Moreover, they can express the various function forms by varying \( \beta \) and \( \gamma \). When \( t = \Delta t \), the boundaries are approximated by \( \alpha_i \) as follows\(^\text{16}\),

\[
q_i(\Delta t) = \alpha_i \left( 1 + \gamma e^{-\beta \Delta t} \right) \approx \alpha_i.
\]

This shows the range of asset \( i \) at time \( \Delta t \) is almost the same as the value of \( \alpha_i \). Therefore, \( \alpha_i \) is called the initial no-trade range. When \( t = T \),

\[
q_i(T) = \alpha_i \left( 1 + \gamma e^{-\beta \Delta t} \right),
\]

and the ranges at time \( T \) are affected by the value of \( \gamma \).

The problem with \( N + 1 \) variables which consist of three kinds of variables \( \alpha_i, \beta \) and \( \gamma \) is solved by the DFO method, and the optimal time-dependent and symmetric no-trade boundaries can be derived.

4.2. Numerical analysis in finite cases

We solve the problems for some combinations of four kinds of time intervals (\( \Delta t = 0.05, 0.1, 0.2, 0.5 \)) and four kinds of time horizons (5, 10, 15, 20 years), and derive the optimal time-dependent ranges by Equation (24). Figure 11 is shown to compare the results for the different time horizons. The horizontal axis is the time to maturity. The graphs above are the results for \( \Delta t = 0.1 \), and the graphs below are for \( \Delta t = 0.5 \). The left graphs are the results for the domestic assets, and the right graphs are for the international assets.

If the time intervals are the same, the no-trade range \( q_i(t) \) is determined by the time to maturity regardless of the length of the time horizon except a domestic bond in 5 year problem. The no-trade ranges are almost the same until the time to maturity is about three years, but the ranges become large as the time to maturity get small.

Figure 12 is shown to compare the results for the different time intervals. The graphs above are the results for 5 year problems, and the graphs below are for 20 year problems.

When the time interval is large, the no-trade range is reduced to avoid the increase in the tracking errors because the investors cannot trade assets except the time points in the discrete-time case.

\(^{16}\)Technically it depends on the values of \( \beta \) and \( \gamma \).
Figure 11: Results for the different time horizons in finite cases

Figure 12: Results for the different time intervals in finite cases
We compare the time-dependent strategy with the constant strategy. Theoretically, the optimal ranges are time-dependent in the finite case, and constant in the infinite case. The no-trade ranges are almost the same when the time point is far from the maturity. It is expected that the initial no-trade ranges for the time-dependent strategy are close to the ranges for the constant strategy as the maturity is long because the ranges near the maturity are averaged over the whole periods. The plots of two values are shown to examine the relationship between them in Figure 13, where the horizontal axis is the constant no-trade range($q_i$), and the vertical axis is the initial no-trade ranges for the time-dependent strategy($\alpha_i$). The points on the 45-degree line could be the same values. Five plots for each time interval in the graph correspond to the time horizons of 5, 10, 15, 20 and 50 years.

Figure 13: Comparison of the time-dependent strategy with the constant strategy

The constant ranges are larger than the initial ranges for the time-dependent strategy. The shorter the maturity is, the larger the constant range is. The longer the maturity is, the closer both values are, and the plots become close to the 45-degree line. As the result, both optimal values are almost the same for 50 year problem. When we need to decide the ‘constant’ no-trade range by the finite horizon model for the time-dependent strategy, we can use the initial no-trade ranges for 50 year problem.

In Section 3, we solve the problem for the constant strategy to use the results in practice despite of the finite time horizon case. However, the objective function value increases more than the time-dependent strategy. We define the deterioration ratios of the objective function values in Equation (28), and show them in Figure 14.
The longer the maturity is, the smaller the deterioration ratio is. This reason is that the range for the constant strategy is closer to the initial range for the time-dependent strategy as the maturity become long. The ratio is about 2% for the case of $\Delta t = 0.1$ and 20 years. Therefore, the constant strategy is much less disadvantaged than the time-dependent strategy as long as the maturity is more than 20 years.

5. Conclusion

In this paper, we derive the optimal solutions numerically for five asset problem with transaction costs and the symmetric no-trade ranges which were not solved in the previous literatures but are needed to solve in practice. In particular, we solve the optimal problem to decide the no-trade ranges according to the parameters used in the GPIF. It is difficult to evaluate the optimal no-trade ranges themselves compared to the ranges used in the GPIF directly because the GPIF does not publish how to decide the ranges. But, the actual weights have moved almost just within the optimal no-trade ranges derived by the DFO method, while the no-trade ranges set by the GPIF are relatively large in comparison with the actual weights. We can show the theoretical methodology and reasoning to decide the optimal no-trade ranges though they are different from the permissible ranges of the GPIF.

We may need to rebalance assets even within the no-trade ranges when the boundary constraints for cash are imposed. We clarify the problem and build the algorithm in the simulation procedure in the discrete-time model. We conduct the sensitivity analysis to examine the influence to the no-trade ranges for the different proportional transaction cost rates, tracking error aversions and cash constraints. In addition, we compare the optimal time-dependent no-trade ranges with the constant no-trade ranges. The numerical results show the possibilities of applying the DFO model to the practical problem determining the symmetric no-trade ranges.
We assume the asset returns are normally distributed in this paper because we develop the model based on Leland[10] and the GPIF use the parameters for the mean-variance model. Various stochastic processes are used to represent the return process such as a time-series model, a jump model, and a model with copula among asset returns. In the future research, we need to adopt the more sophisticated model, and solve the optimal problem by the DFO approach.

References