A Hybrid Simulation/Tree Stochastic Optimization Model for Dynamic Asset Allocation

Norio Hibiki

Faculty of Science and Technology, Keio University
3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan
hibiki@ae.keio.ac.jp

Abstract

Asset allocation decisions are critical for investors with diversified portfolios. Institutional investors must manage their strategic asset mix over time to achieve favorable returns subject to various uncertainties, policy and legal constraints, and other requirements. In order to determine the asset mix explicitly, one may use a multi-period portfolio optimization model. The concept of scenarios is typically employed for modeling random parameters in multi-period stochastic programming (MSP) models, and scenarios are constructed via a tree structure. Another approach for developing dynamic investment strategies, which offers an alternative to stochastic programming, is the dynamic stochastic control. Recently, an alternative stochastic programming model with simulated paths was proposed by Hibiki (2001b, 2001c). Scenarios are constructed via a simulated path structure. The advantage of simulated paths comparing to scenario trees is higher accuracy of description of uncertainties associated with asset returns. In addition, we can make conditional decisions in this framework similarly to a scenario tree model. This model is called a hybrid model. The model is formulated as a linear programming model, which can be easily implemented and efficiently solved using sophisticated mathematical programming software.

The previous papers (Hibiki 2001b, 2001c) do not have enough results to show the features of the hybrid model. In this paper, we develop the general formulation for several investment strategies, and highlight its features and properties using some numerical tests.

We explain the concept, formulation and typical numerical examples of the hybrid model, the scenario generating process, and the procedure of generating extended decision tree. We present some numerical tests for various numbers of branching trees, various numbers of simulated paths, degree of the sampling error, and different investment strategies. Moreover, we show the compact representation form, which is equivalent to the original form but is more efficient from computational point of view.
1 Introduction

Rational investors maximize the expected utility of return from their investment portfolio, or minimize the risk exposure of return, subject to the required expected return. They must decide on their optimal portfolio in securities in order to meet their satisfaction. This paper discusses optimal dynamic investment policies for investors, who make the investment decision in each asset category over time. This problem is called “dynamic asset allocation”.

Asset allocation decisions are critical for investors with diversified portfolios. Institutional investors must manage their strategic asset mix over time to achieve favorable returns, in presence of uncertainties, and subject to various legal constraints, policies, and other requirements. In order to determine the asset mix explicitly, a multi-period portfolio optimization model can be used.

It is critical for stochastic modeling to handle uncertainties and investment decisions appropriately. The decisions have to be independent from knowledge of actual paths that will occur. Thus, we must define a set of decision variables and a set of constraints to prevent the optimization model from being solved anticipating the event in the future. This is called non-anticipativity condition in the stochastic programming model. In addition, we need a sufficient number of paths to get a better accuracy with respect to the future possible events.

The concept of scenarios is typically employed for modeling random parameters in multi-period stochastic programming (MSP) models. Scenarios are constructed via a tree structure (see Mulvey and Ziemba, 1995 and 1998, for a detail discussion). This model is based on the expansion of the decision space, taking into account the conditional nature of the scenario tree. Conditional decisions are made at each node, subject to the modeling constraints. To ensure that the constructed representative set of scenarios covers the set of possibilities to a sufficient degree, the number of decision variables and constraints in the scenario tree may grow exponentially.

Another approach for developing dynamic investment strategies that offers an alternative to stochastic programming, is dynamic stochastic control. The basic framework of this model was originally proposed by Merton (1969) and Samuelson (1969). In general, stochastic control forms a mesh over the state space, instead of discretizing the scenarios. However, in order to construct detailed discretizations, one needs a lot of computational resources.

Recently, an alternative stochastic programming model using simulated paths was proposed by Hibiki (2001b and 2001c). Scenarios are constructed via a simulated path structure. We can generate sample paths associated with asset returns using the Monte Carlo simulation method. Therefore, the advantage of simulated paths compared to a scenario tree gives a better descriptive accuracy of the uncertainties associated with asset returns. In addition, we can make conditional decisions in this framework as with a scenario tree model. This model is called a “hybrid model”, because it can describe the accuracy of uncertainties and make the conditional decisions simultaneously \(^1\). The model is formulated as a linear programming model, which can be easily

\(^1\)Hibiki (2000 and 2001a) developed the simulated path model. The model also requires simulated paths to have the accuracy of uncertainties, but it cannot make conditional decisions. The hybrid model is allowed to
implemented and efficiently solved using sophisticated mathematical programming software. Previous papers (Hibiki, 2001b and 2001c) do not have enough results to show the features of the hybrid model. This paper develops the general formulation for several investment strategies, and highlight its features and properties by using numerical tests.

The paper is organized as follows. Section 2 presents the concept, formulation and typical numerical examples of the hybrid model. It is followed by a demonstration of the scenario generating process in Section 3 and an explanation of the procedure of generating extended decision trees in Section 4. Section 5 presents some numerical tests for various numbers of branching trees and simulated paths, degrees of the sampling error, and different investment strategies. Section 6 shows the compact representation form, which is equivalent to the form in Section 2, but is more efficient from computational point of view. Section 7 provides some concluding remarks and outlines our future research.

2 The hybrid model

2.1 Modelling for conditional decisions

An asset return is a random parameter, and its process is expressed by a stochastic differential equation, or a time series model. We can sample simulated paths of each asset return on each simulation trial. An example of simulated paths is shown as in the left side of Figure 1.

Hibiki (2001b and 2001c) develop the hybrid model in a multi-period optimization framework. Discrete values of asset return are generated by Monte Carlo simulation to describe the uncertainties more accurately than would the scenario tree, as in the right side of Figure 1. However, expand the decision space and to make conditional decisions as used in the scenario tree model. The simulated path model is a special version of the hybrid model.
we have to define a different decision rule in the simulated path approach from the one in the scenario tree approach. Consider a path $i$ of asset returns in a set of the simulated paths. The returns at time $t+1$ are known on the path $i$ at time $t$ if returns on the path $i$ occurs in the future. If each decision is made on each path, the model can be solved by anticipating the future. Therefore, we must define a set of decision variables and a set of constraints to prevent the optimization model from being solved by anticipating the event in the future.

Conditional decisions can keep the non-anticipativity condition in the simulated path approach. We make several bundles of simulated paths at each time, and we have a fixed strategy (decision rule) for risky assets at each bundle. The bundles are called “fixed-decision nodes”, and the decision tree is generated where conditional decisions are made at each node. This tree is called the “extended decision tree” to distinguish from the scenario tree. Under the decision rule, the decisions of the risky assets are fixed at each bundle. However, the decisions on cash can be path-dependent, because cash return is based on the interest rate, which is risk-free at each time when decision is made.

Suppose 12 simulated paths over three periods are represented as in the left side of Figure 1. For simplicity, we assume we make three bundles of 12 paths at time 1, and more two bundles of paths within each bundle at time 2. In total, we have six bundles at time 2. These bundles are shown as in the left side of Figure 2. We call it “3-2” branching tree. The right side of Figure 2 is described as the tree structure, which is called the “extended decision tree”, and shows the same structure as the left one.

We can select a fixed strategy for risky assets, such as fixed-proportion strategy, fixed-value strategy, fixed-unit strategy, and so on. Even if we select investment units as decision variables, we do not have to fix investment units at each node. If we define the function of the decision rule associated with the investment units, we can invest the different units on each path through a
node. This is called the “investment unit function”. Moreover, we can describe other strategies such as fixed-proportion strategy and fixed-value strategy by using this function.

2.2 Model formulation

We invest in $n$ risky assets and cash. The investment is made at time 0 (present), and time $T$ is the planning horizon. We determine the asset mix by using two kinds of measures; the expected terminal wealth and the first-order lower partial moment ($LPM_1$) of terminal wealth (see Bawa and Lindenberg, 1977, Harlow, 1991). The former corresponds to the return measure and the latter corresponds to the risk measure. The lower partial moment is a downside risk measure, and expresses the tail risk of the relevant distribution of wealth below target.

Computationally, the LPM for an empirical (discrete) distribution of terminal wealth, $W_T^{(i)}$, with a target wealth, $W_G$, is described by: \(^2\)

$$LPM_k = \frac{1}{I} \sum_{i=1}^{I} |W_T^{(i)} - W_G|^k$$

(1)

where $I$ is the number of samples, and $|a|_- = \max(-a, 0)$. The risk measure corresponds to the first-order LPM for $k = 1$. If we select the strategy that has a linear investment unit function, we can formulate as a linear programming problem, and solve a large-scale problem easily in practical use. The notations in this model are as follows.

(1) Sets

$S_t$: set of fixed-decision nodes at time $t$ ($s \in S_t$)

$V_s^t$: set of paths including any fixed-decision node $s$ at time $t$ ($i \in V_s^t$)

(2) Parameters

$I$: number of simulated paths

$\rho_{jt}$: price of risky asset $j$ at time $t$, ($j = 1, \ldots, n$)

$\rho_{jt}^{(i)}$: price of risky asset $j$ of path $i$ at time $t$, ($j = 1, \ldots, n$; $t = 1, \ldots, T$; $i = 1, \ldots, I$)

$r_{0}$: interest rate in period 1 (the rate at time 0 is used).

$r_{i-1}^{(i)}$: interest rate of path $i$ in period $t$ (the rate at time $t-1$ is used), ($t = 2, \ldots, T$; $i = 1, \ldots, I$)

$W_0$: initial wealth

$W_G$: target terminal wealth

$W_E$: required expected terminal wealth

$\gamma$: risk-averse coefficient

\(^2\)The LPM for a continuous distribution of terminal wealth $\tilde{W}_T$ is described as follows:

$$LPM_k = \int_{-\infty}^{W_G} (W_G - \tilde{W}_T)^k f(\tilde{W}_T) d\tilde{W}_T$$
(3) Decision variables

- $z_{j0}$: investment unit for asset $j$ at time 0 ($j = 1, \ldots, n$)
- $z_{jt}^s$: base investment unit for asset $j$ of node $s$ at time $t$ ($j = 1, \ldots, n; t = 1, \ldots, T-1; s \in S_t$).
- $v_0$: cash at time 0
- $v_{it}^i$: cash of path $i$ at time $t$ ($t = 1, \ldots, T-1; i = 1, \ldots, I$)
- $q_{it}^i$: shortfall below target terminal wealth of path $i$ at the planning horizon, ($i = 1, \ldots, I$)

2.2.1 Investment strategies with investment unit functions

We define the investment unit function as follows $^3$.

$$h_{it}^i(z_{it}^s) = a_{it}^i z_{it}^s$$

where $a_{it}^i$ is the investment unit parameter. To keep non-anticipativity condition, $a_{it}^i$ must be independent on the rate of returns of path $i$ after time $t$. We consider three kinds of investment strategies with investment unit functions.

1. **Fixed-unit strategy**: $h_{it}^i(z_{it}^s) = z_{it}^s$

   All risky assets have the same investment units on any path at each node, respectively. However, cash is different in each path.

2. **Fixed-value strategy**: $h_{it}^i(z_{it}^s) = \left( \frac{\rho_{it}}{\rho_{jt}} \right) z_{jt}^s$

   All risky assets have the same investment values on any path at each node, respectively. However, cash is different in each path $^4$.

3. **Fixed-proportion strategy**: $h_{it}^i(z_{it}^s) = \left( \frac{W_{it}^i}{\rho_{jt}} \right) z_{jt}^s$

   All risky assets and cash have the same investment proportions on any path at each node, respectively.

Constraints are linear in the fixed-unit strategy and fixed-value strategy. But constraints are non-convex in the fixed-proportion strategy because $W_{it}^i$ is a function of decision variables $^5$.

2.2.2 Formulation

We use two kinds of objective functions. The first type is the “expected wealth and risk” (ER) function, which minimizes risk ($LPM_1$) subject to the constraint where expected terminal wealth

$$h_{it}^i(z_{it}^s, k) = \sum_{k=1}^K a_{jt,k}^i z_{jt,k}^s$$

$^3$It can be also defined using multiple ($K$) base investment units $z_{jt,k}^s$.

$^4$This strategy is a kind of contrarian investment strategies, because an investment unit tends to be decreased when price goes up, and tends to be increased when price goes down.

$^5$The non-convex program is solved approximately by the iterative method in Section 5.4. However, details have been omitted due to lack of space.
The following objective function is equivalent to Equations (3) – (5).

\[ \text{Minimize} \quad \frac{1}{I} \sum_{i=1}^{I} q^{(i)} \]  
\[ \text{subject to} \quad W_T^{(i)} + q^{(i)} \geq W_G, \quad (i = 1, \ldots, I) \]  
\[ q^{(i)} \geq 0, \quad (i = 1, \ldots, I) \]  
\[ \frac{1}{I} \sum_{i=1}^{I} W_T^{(i)} \geq W_E \]  \hspace{1cm} (6)

\[ \text{Minimize} \quad \frac{1}{I} \sum_{i=1}^{I} W_T^{(i)} - \gamma \left( \frac{1}{I} \sum_{i=1}^{I} q^{(i)} \right) \]  
\[ \text{subject to} \quad \text{Equations (4) and (5)} \]

We show the hybrid model with the ER-type objective function as follows.

\[ \text{Minimize} \quad \frac{1}{I} \sum_{i=1}^{I} q^{(i)} \]  
\[ \text{subject to} \]  
\[ \sum_{j=1}^{n} \rho_{j0} z_{j0} + v_0 = W_0 \]  \hspace{1cm} (9)
\[ \sum_{j=1}^{n} \rho_{j1} z_{j0} + (1 + r_0)v_0 = \sum_{j=1}^{n} \rho_{j1} h^{(i)}(z_{j1}) + v_1^{(i)}, \quad (s \in S_1; \ i \in V_1^s) \]  \hspace{1cm} (10)
\[ \sum_{j=1}^{n} \rho_{jt}^s h^{(i)}(z_{jt}) + (1 + r_{t-1}) v_{t-1}^{(i)} = \sum_{j=1}^{n} \rho_{jt}^s h^{(i)}(z_{jt}) + v_t^{(i)}, \quad (t = 2, \ldots, T - 1; \ s \in S_t; \ i \in V_t^s) \]  \hspace{1cm} (11)
\[ W_T^{(i)} = \sum_{j=1}^{n} \rho_{jT}^s h^{(i)}(z_{jT-1}) + (1 + r_{T-1}^{(i)}) v_{T-1}^{(i)}, \quad (s' \in S_{T-1}; \ i \in V_{T-1}^{s'}) \]  \hspace{1cm} (12)
\[ \frac{1}{I} \sum_{i=1}^{I} W_T^{(i)} \geq W_E \]  \hspace{1cm} (13)
\[ W_T^{(i)} + q^{(i)} \geq W_G, \quad (i = 1, \ldots, I) \]  \hspace{1cm} (14)
\[ z_{j0} \geq 0, \quad (j = 1, \ldots, n) \]  \hspace{1cm} (15)
\[ z_{jt}^s \geq 0, \quad (j = 1, \ldots, n; \ t = 1, \ldots, T - 1; \ s \in S_t) \]  \hspace{1cm} (16)
\[ v_0 \geq 0 \]  \hspace{1cm} (17)
\[ v_t^{(i)} \geq 0, \quad (t = 1, \ldots, T - 1; \ i = 1, \ldots, I) \]  \hspace{1cm} (18)

\[ \text{The following objective function is equivalent to Equations (3) – (5).} \]

\[ \text{Minimize} \quad \frac{1}{I} \sum_{i=1}^{I} \left| W_T^{(i)} - W_G \right| \]
\[ q^{(i)} \geq 0, \quad (i = 1, \ldots, I) \]  

\( W^{(i)}_T \) is the terminal wealth for time \( T \) and path \( i \) \((i = 1, \ldots, I) \). We denote \( s' \) in Equation (11) to be the decision node at time \( t - 1 \) connected with the node at time \( t \). The values of both sides of Equations (10) and (11) show the wealth of path \( i \) at time \( t \).

### 2.3 Numerical Examples

This section reports results of numerical examples. In the first example, four assets (stock, bond, convertible bond (CB) and cash) are solved over four periods, using a 5-4-3 branching tree model (5-4-3 branching means 5 branching at time 1, 4 branching at time 2, and 3 branching at time 3). The number of simulated paths is 10,000. The number of constraints except non-negative constraints, and the number of decision variables are about 50,000.

Initial prices of stock, bond, and CB can be assumed to be 1 without loss of generality. The initial call rate is 0.44%. The initial wealth is 100 million Japanese yen, and the target terminal wealth is also 100 million Japanese yen. We select the fixed-unit strategy and generate rates of returns using summary statistics shown in Table 1 in the following section.

We show the results in the case of \( W_E = 10,340 \) (unit: ten thousand Japanese yen). Optimal units for the four assets are obtained as in Figure 3. Due to large number of cash variables, we show only the average cash value for each node, \( \bar{v}_i \).

---

7The examples here were solved using NUOPT (version 5.1.0a) — mathematical programming software developed by Mathematical System, Inc. — on Windows 2000.
Five kinds of conditional optimal investment units can be obtained at time 1. We can also obtain four kinds at time 2 subject to the corresponding node at time 1 and three kinds at time 3 subject to the corresponding node at time 2. Figure 4 shows the efficient frontier (right side) and the cumulative discrete distribution (left side), which help investors decide the investment policy. We describe three kinds of cumulative distributions for $W_E = 10, 280$, $10, 340$, $10, 385$. The larger the required expected terminal wealth, the fatter the tail of the distribution is below target terminal wealth ($W_G = 10, 000$).

![figure 4](image)

**Figure 4:** Efficient frontier and cumulative discrete distribution of the terminal wealth

### 2.4 Computation time

Figure 5 shows computation time for 15 kinds of numbers of path and 14 kinds of $\gamma$, which are solved using the EU model. It takes about two minutes to solve 15,000 paths problem. It has about 75,000 constraints and about 75,000 decision variables, but it has a very sparse matrix, so we can solve such large problems very fast. Though the number of paths ranges up to 15,000, the relation between computation time and the number of paths may be approximately linear.

![figure 5](image)

**Figure 5:** Computation time
3 Scenario Generation

In general, scenarios associated with asset returns are generated according to the stochastic differential equations or time-series models. Mulvey and Thorlacius (1998) use Towers Perrin’s scenario generation system, “CAP: Link” to solve a multi-period stochastic programming problem for pension funds. A scenario system is based on a cascading set of stochastic differential equations. The basic stochastic differential equations are identical in each country, although the parameters reflect unique characters of each particular economy. Economic variables such as interest rate, inflation rate, actual yield, exchange rate and stock return, are expressed as stochastic differential equations. For example, short-term and long-term interest rates are expressed as a variant of a two-factor Brennan-Schwartz model. The Russel-Yasuda model[see Carino, D.R et al. (1998a,1998b and 1998c)] used for the ALM of casualty insurance company, generates scenarios whose returns are created from a factor model that incorporates dependence between periods.

If a set of scenarios is constructed by a tree structure, the problem size may grow exponentially. Although we need to describe asset returns appropriately by a moderate number of scenarios, the detail implementation method has not been expressed in those papers(Mulvey and Thorlacius, 1998, Carino et al., 1998a, 1998b and 1998c). The hybrid model needs a sample path structure generated by Monte Carlo simulation as in the left side of Figure 1. If scenario generation model is selected, we can generate such sample paths easily by a standard procedure of Monte Carlo simulation technique. It is important which model is selected because the optimal solutions change according to the model.

However, the main aim of this paper is to explain how to construct the optimization model with simulated paths and to clarify the feature of the method. Then we use the following simple procedure with the statistics associated with asset returns(expected rate of return, standard deviation and correlation matrix of rate of return) to generate scenarios of rates of returns of n risky assets and call rate.

The rate of return $\mu_{jt}^{(i)}$ is generated as follows, where asset 0 ($j = 0$) corresponds to call rate.

1. The rate of return of asset $j$ in period $t$ is normally distributed with mean $\overline{\mu}_{jt}$ and standard deviation $\sigma_{jt}$, and it is generated by:
   $$\mu_{jt}^{(i)} = \overline{\mu}_{jt} + \sigma_{jt} \varepsilon_{jt}^{(i)};$$
   where $\varepsilon_{jt}^{(i)}$ is a random sample from a multi-variate standardized normal distribution.

2. The random variable $\varepsilon_{jt}$ $(j = 0, \ldots, n; t = 1, \ldots, T)$ follows that $\varepsilon_{jt} \sim N(0, \Sigma)$,
   where $\Sigma$ is $(n + 1)T \times (n + 1)T$ correlation matrix.
   $\mu_{0t}^{(i)}$ is the change rate of call rate. The call rate $r_{1t}^{(i)}$ is calculated by:
   $$r_{1t}^{(i)} = r_0 \times \left(1 + \mu_{01}^{(i)}\right),$$
Table 1 shows the summary statistics calculated by the available market data; Nikko stock performance index (TSE 1), Nikko bond performance index, Nikko CB performance index, and call rate.

Table 1: Summary statistics

<table>
<thead>
<tr>
<th></th>
<th>cash 1</th>
<th>cash 2</th>
<th>cash 3</th>
<th>cash 4</th>
<th>stock 1</th>
<th>stock 2</th>
<th>stock 3</th>
<th>stock 4</th>
<th>bond 1</th>
<th>bond 2</th>
<th>bond 3</th>
<th>bond 4</th>
<th>CB 1</th>
<th>CB 2</th>
<th>CB 3</th>
<th>CB 4</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Exp. Value</strong></td>
<td>-0.087</td>
<td>-0.091</td>
<td>-0.073</td>
<td>-0.073</td>
<td>0.848</td>
<td>0.867</td>
<td>0.843</td>
<td>0.858</td>
<td>0.625</td>
<td>0.625</td>
<td>0.645</td>
<td>0.623</td>
<td>0.780</td>
<td>0.780</td>
<td>0.866</td>
<td>0.809</td>
</tr>
<tr>
<td><strong>St. Dev.</strong></td>
<td>0.786</td>
<td>0.784</td>
<td>0.778</td>
<td>0.779</td>
<td>5.571</td>
<td>5.582</td>
<td>5.590</td>
<td>5.591</td>
<td>1.372</td>
<td>1.372</td>
<td>1.353</td>
<td>1.233</td>
<td>3.543</td>
<td>3.541</td>
<td>3.538</td>
<td>3.525</td>
</tr>
</tbody>
</table>

4 Procedure of generating extended decision tree

4.1 Classifying method

Suppose 12 simulated paths and nodes over three periods are represented as in the left side of Figure 2. The bundling procedure is illustrated schematically in this figure. First, we generate 12 paths associated with asset returns over the planning period. Next, 12 paths are classified into three nodes of four paths in period 1, named node A, B, and C. The conditional fixed-decisions are made at each node, respectively. Finally, the same procedure is carried out in every node at time 1 in period 2. Specifically, four paths through node A are classified into two nodes in period 2. We have two paths in each node. Similarly, four paths through node B, and four paths through node C are classified into two nodes. We have two paths in each node from node B, and have two paths from node from node C. We have six kinds of conditional decisions at time 2.

We may use any classifying method, but this paper focuses on in particular, as follows.

(1) Sequential clustering method (SQC method)

This method is applied to the data set of simulated paths over the planning period by using the well-known hierarchical clustering method \(^8\) in each period sequentially. Generated clusters represent the fixed-decision nodes. The method is implemented based on similarities calculated by distances between sampled return vectors.

(2) Portfolio based clustering method (PBC method)

This method is applied to the wealth of path i at time t which is calculated by any portfolio over the planning period. We can use any portfolio, such as an equally weighted portfolio, an optimal portfolio derived by solving the simulated path model \(^9\), and so on. But it is dependent on the model which portfolios are appropriate to the model. Therefore, we need to compare some portfolios to solve the model.

\(^8\) We use the Ward method, which is one of the hierarchical clustering methods. Other method are the nearest neighbor method, the furthest neighbor method, the group average method, and so on. It is said that the Ward method is superior to other methods in practical use. See Tanaka, Y., and Wakimoto, K. (1983)

\(^9\) This idea was originally developed in Bogentoft, Romeijn and Uryasev (2001).
We compare three methods using numerical examples:

1. the SQC method;
2. the PBC method with an equal-weight portfolio (W-PBC method);
3. the PBC method with an optimal portfolio for the simulated path model (S-PBC method).

We implement the PBC method so that the number of paths through node is equal.

The number of paths is 3,000, and 100 kinds of random seeds are used. The ER model with the fixed-unit strategy is solved for 5 kinds of terminal wealth ($W_E$): $W_E = 10, 280, 10, 310, 10, 340, 10, 370, 10, 385$.

Table 2: Comparison of three methods; statistics of risk value

<table>
<thead>
<tr>
<th></th>
<th>SQC</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_E$</td>
<td>10, 280</td>
<td>10, 310</td>
<td>10, 340</td>
<td>10, 370</td>
<td>10, 385</td>
</tr>
<tr>
<td>Average</td>
<td>6.10</td>
<td>16.40</td>
<td>33.62</td>
<td>69.33</td>
<td>110.10</td>
</tr>
<tr>
<td>St.Dev.</td>
<td>0.84</td>
<td>1.82</td>
<td>3.61</td>
<td>10.90</td>
<td>22.73</td>
</tr>
<tr>
<td>Maximum</td>
<td>8.02</td>
<td>20.39</td>
<td>42.99</td>
<td>108.36</td>
<td>191.48</td>
</tr>
<tr>
<td>75% pt.</td>
<td>6.75</td>
<td>17.68</td>
<td>36.28</td>
<td>75.96</td>
<td>120.12</td>
</tr>
<tr>
<td>Median</td>
<td>6.07</td>
<td>16.47</td>
<td>33.49</td>
<td>67.37</td>
<td>105.27</td>
</tr>
<tr>
<td>25% pt.</td>
<td>5.53</td>
<td>15.24</td>
<td>31.07</td>
<td>60.91</td>
<td>93.81</td>
</tr>
<tr>
<td>Minimum</td>
<td>3.84</td>
<td>10.98</td>
<td>23.40</td>
<td>50.25</td>
<td>75.09</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>W-PBC</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_E$</td>
<td>10, 280</td>
<td>10, 310</td>
<td>10, 340</td>
<td>10, 370</td>
<td>10, 385</td>
</tr>
<tr>
<td>Average</td>
<td>5.19</td>
<td>13.83</td>
<td>27.57</td>
<td>54.27</td>
<td>86.24</td>
</tr>
<tr>
<td>St.Dev.</td>
<td>0.65</td>
<td>1.28</td>
<td>2.24</td>
<td>6.32</td>
<td>14.75</td>
</tr>
<tr>
<td>Maximum</td>
<td>7.41</td>
<td>18.30</td>
<td>35.99</td>
<td>81.87</td>
<td>136.64</td>
</tr>
<tr>
<td>75% pt.</td>
<td>5.66</td>
<td>14.54</td>
<td>28.95</td>
<td>57.93</td>
<td>93.44</td>
</tr>
<tr>
<td>Median</td>
<td>5.11</td>
<td>13.74</td>
<td>27.61</td>
<td>53.74</td>
<td>85.30</td>
</tr>
<tr>
<td>25% pt.</td>
<td>4.74</td>
<td>12.99</td>
<td>26.05</td>
<td>50.71</td>
<td>76.97</td>
</tr>
<tr>
<td>Minimum</td>
<td>3.51</td>
<td>10.67</td>
<td>21.52</td>
<td>39.93</td>
<td>54.77</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>S-PBC($\gamma = 1.5$)</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_E$</td>
<td>10, 280</td>
<td>10, 310</td>
<td>10, 340</td>
<td>10, 370</td>
<td>10, 385</td>
</tr>
<tr>
<td>Average</td>
<td>1.42</td>
<td>6.01</td>
<td>13.91</td>
<td>28.18</td>
<td>41.05</td>
</tr>
<tr>
<td>St.Dev.</td>
<td>0.36</td>
<td>0.78</td>
<td>1.33</td>
<td>2.75</td>
<td>4.95</td>
</tr>
<tr>
<td>Maximum</td>
<td>2.27</td>
<td>7.60</td>
<td>16.95</td>
<td>35.73</td>
<td>54.21</td>
</tr>
<tr>
<td>75% pt.</td>
<td>1.66</td>
<td>6.55</td>
<td>15.03</td>
<td>29.94</td>
<td>44.61</td>
</tr>
<tr>
<td>Median</td>
<td>1.44</td>
<td>6.05</td>
<td>14.00</td>
<td>28.06</td>
<td>40.33</td>
</tr>
<tr>
<td>25% pt.</td>
<td>1.19</td>
<td>5.52</td>
<td>13.05</td>
<td>25.95</td>
<td>37.39</td>
</tr>
<tr>
<td>Minimum</td>
<td>0.61</td>
<td>4.02</td>
<td>10.91</td>
<td>22.25</td>
<td>31.63</td>
</tr>
</tbody>
</table>

Table 2 shows the summery of statistics of risk value for three methods. The simulated path model is solved for an appropriate risk averse coefficient $\gamma$ in the S-PBC method.

As Table 3 shows, when using $\gamma = 1.5$, the S-PBC method is much better than the other two methods. The reason is that three average risk values that have been calculated by using 100 kinds of random seeds are the smallest for $\gamma = 1.5$. In fact, both the average risk and standard deviation of risk of the S-PBC method are smaller than the SQC method and the W-PBC method. Moreover, the maximum risk of the S-PBC method is smaller than the minimum risk of other methods. This means the distribution derived from the S-PBC method dominates the others. The reason is that the set of wealth calculated by the simulated path model is more similar to the set of wealth by the hybrid model than others, although the levels of wealth themselves differ.

### 4.2 Choice of risk-averse coefficient of the simulated path model for the S-PBC method

Table 3 shows the average risk of the hybrid model calculated using 100 kinds of random seeds for 13 kinds of risk-averse coefficients in the S-PBC method. Average risk values in the case
of $\gamma = 1.5$ are the smallest for $WE = 10,280,10,310,10,340$, and average risk values in the case of $\gamma = 1.25$ is the smallest for $WE = 10,370,10,385$. When we solve the problem with $\gamma$, which ranges from 1 to 10, we have the similar values of average risk. It seems that $\gamma$ is not sensitive to the average risk within this range, although the risk values of simulated path model and associated portfolio differ according to $\gamma$. Therefore, we use $\gamma = 1.5$ of the S-PBC method for numerical analysis in Section 5 to make an extended decision tree.

### Table 3: Average risk for various risk averse coefficient $\gamma$

<table>
<thead>
<tr>
<th>Hybrid model $(WE)$</th>
<th>$\gamma$</th>
<th>10,280</th>
<th>10,310</th>
<th>10,340</th>
<th>10,370</th>
<th>10,385</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>1.941</td>
<td>7.141</td>
<td>16.007</td>
<td>32.080</td>
<td>47.550</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>1.917</td>
<td>7.061</td>
<td>15.861</td>
<td>31.708</td>
<td>46.801</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1.547</td>
<td>6.287</td>
<td>14.455</td>
<td>29.045</td>
<td>42.426</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.530</td>
<td>6.208</td>
<td>14.259</td>
<td>28.718</td>
<td>41.806</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.540</td>
<td>6.253</td>
<td>14.346</td>
<td>28.935</td>
<td>42.116</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.475</td>
<td>6.139</td>
<td>14.161</td>
<td>28.669</td>
<td>41.645</td>
<td></td>
</tr>
<tr>
<td>1.75</td>
<td>1.443</td>
<td>6.073</td>
<td>14.040</td>
<td>28.416</td>
<td>41.334</td>
<td></td>
</tr>
<tr>
<td>1.50</td>
<td>1.416</td>
<td>6.012</td>
<td>13.907</td>
<td>28.183</td>
<td>41.052</td>
<td></td>
</tr>
<tr>
<td>1.25</td>
<td>1.431</td>
<td>6.048</td>
<td>13.982</td>
<td>28.163</td>
<td>40.924</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.556</td>
<td>6.325</td>
<td>14.448</td>
<td>28.945</td>
<td>42.104</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>1.826</td>
<td>6.894</td>
<td>15.396</td>
<td>30.274</td>
<td>43.763</td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>2.646</td>
<td>8.631</td>
<td>18.354</td>
<td>35.098</td>
<td>51.671</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>5.787</td>
<td>15.237</td>
<td>30.629</td>
<td>61.691</td>
<td>99.829</td>
<td></td>
</tr>
</tbody>
</table>

## 5 Numerical analysis

We analyze the hybrid model numerically. We test the following cases using a 5-4-3 branching tree, except in Case 1.

1. Case 1: Comparison of the results for various numbers of branching trees
   - 12 kinds of N-N-N branching trees are compared. ($N = 2, 3, \ldots, 13$)

2. Case 2: Comparison of the results for various numbers of simulated paths
   - Six kinds of simulated paths are compared, such as $I = 1,000, 3,000, 5,000, 7,000, 10,000, 15,000$.

3. Case 3: Evaluation of sampling error
   - We use 100 kinds of random seeds to evaluate sampling error. We solve the hybrid model for five kinds of the required expected terminal wealth and six kinds of the numbers of simulated paths.

4. Case 4: Comparison of different investment strategies
   - We solve the hybrid model for three different investment strategies: fixed-unit strategy, fixed-value strategy, and fixed-proportion(ratio) strategy.

### 5.1 Case 1: Comparison of the results for various numbers of branching trees

Figure 6 shows the efficient frontier for various numbers of branching trees. The number of branching trees affects the result. As the number of branching trees increases, the efficient frontier moves upwards, because more a flexible way to invest can be selected. This requires some degree of concentration, as we need enough paths to describe the uncertainty if the large branching tree is required.

Figure 7 shows the optimal portfolio at time 0 for 3-3-3, 8-8-8, 13-13-13 branching trees. As the number of branching tree increases, we tend to invest more risky assets at time 0, little by
little. This can be done because more flexible investments can control risk, even if more risky assets are invested at time 0.

Figure 6: Efficient frontier for various numbers of branching tree

Figure 7: Optimal portfolio at time 0 for various numbers of branching tree

5.2 Case 2: Comparison of the results for various numbers of simulated paths

Figure 8: Efficient frontier for various numbers of simulated paths

Figure 8 shows the efficient frontier for various numbers of simulated paths. As the number of simulated paths decreases, the efficient frontier moves upwards because the problem with a smaller number of paths is over-evaluated in comparison with the problem with a larger number of paths. The better setting is to have larger numbers of branching trees and simulated paths,
but this would be impossible to solve due to the limit of the computer resources. The appropriate combination of the number of branching trees and number of simulated paths is still an open question in this paper.

Figure 9 shows the optimal portfolio at time 0 for various numbers of simulated paths. As the number of simulated paths decreases, we intend to invest more in risky assets. This reason for this is that it is easy to control risk even if we can invest in riskier assets.

Figure 9: Optimal portfolio at time 0 for various numbers of simulated paths

5.3 Case 3: Evaluation of Sampling error

Five figures in Figure 10 show the relationship between the number of simulated paths and risk(LPM$_1$), and a lower right figure shows average risk for five kinds of $W_E$. As the number of simulated paths increases, so does risk value. This means that the efficient frontier moves downwards as in Figure 8, and it does not converge in 15,000 paths because the description of uncertainty is not enough. However, the change in slope decreases little by little. Therefore, it is expected that the average risk value converges if the number of simulated paths increases further. Standard deviation does not become small because the average risk value does not converge in 15,000 paths. But it seems that variability becomes relatively small. The issue of sampling error is serious for the hybrid model. We need to decrease the number of branching trees to reduce sampling error in the 10,000-path problem. But the optimum number of paths also remains an open question in this case.

Figure 10: The relationship between the number of simulated path and risk(LPM$_1$)
5.4 Case 4: Comparison of different investment strategies

Figure 11 shows the efficient frontier for different investment strategies. In this example, the fixed-proportion strategy dominates the other two strategies (the fixed-unit and the fixed-value strategies). We need to hold cash after time 1 to execute transactions for the fixed-unit and the fixed-value strategies in the simulated path approach. On the other hand, we do not always have to hold cash for the fixed-proportion strategy. In the case that $\gamma$ is relatively larger, three strategies have almost the same values. This is due to the fact cash is held because of risk reduction, as can be verified by studying Figure 12, which shows the optimal average proportions at each time for different investment strategies.

In this example, the fixed-value strategy dominates the fixed-unit strategy in the case that the range of $\gamma$ is from 0.2 to 1. This reason is because Table 1 has some negative serial correlation coefficients, and fixed-value strategy is a contrarian strategy\(^{10}\). On the other hand, the maximum expected wealth of fixed-value strategy is smaller than that of the fixed-unit strategy. Then the fixed-unit strategy dominates the fixed-value strategy when $\gamma$ is small.

---

\(^{10}\)A contrarian strategy is one of the investment strategies that investment unit is decreased when price goes up, and increased when price goes down. When the serial correlation is positively large, the fixed-unit strategy tends to dominate the fixed-value strategy. Details are omitted due to lack of space.
6 Compact representation

Here, we discuss the equivalent formulation to decrease the problem size and its computation time. According to the modeling structure, the number of cash decision variables depends on the set of paths and periods, and this leads to a large-scale problem because of large number of paths. On the other hand, the number of decision variables of any risky asset depends on only the number of decision nodes. The original formulation is an easy-to-understand hybrid model, but it does not make effective use of the decision rule with fixed strategy for risky assets at each node in order to reduce the problem size. Because of this, we develop an equivalent and more compact formulation by eliminating cash variables. This does not mean that cash is excluded from the asset allocation decision; we transform the original form into two kinds of formulations: primal compact representation and dual compact representation. Due to lack of space, we omit
the description of the formulation of primal form \(^{11}\). Dual compact representation is the dual form of the primal compact representation. For simplicity, we denote investment units by \(h^{(i)}(z_{jt}^{*}) = z_{jt}^{*}\). The dual compact formulation of the hybrid model is described as follows:

\[
\begin{align*}
\text{Maximize} & \quad - W_0 \lambda_0 - \sum_{i=1}^{I} \sum_{t=1}^{T-1} F_t^{(i)} \lambda_t^{(i)} + \sum_{i=1}^{I} \left( W_G - F_T^{(i)} \right) \lambda_T^{(i)} + \left( W_E - F_T \right) \omega \\
\text{subject to} & \quad -\rho_j \lambda_0 + \sum_{i=1}^{I} \sum_{t=1}^{T} \eta_{j0t} \lambda_t^{(i)} + \eta_{j0T} \omega \leq 0, \quad (j = 1, \ldots, n) \\
& \quad - \sum_{i_k \in V^{sk}_k} r_{jk}^{(i_k)} \lambda_k + \sum_{i_k \in V^{sk}_k} \sum_{t=k+1}^{T} \eta_{jkt}^{(i_k)} \lambda_t^{(i_k)} + \eta_{jkT}^{sk} \omega \leq 0, \quad (j = 1, \ldots, n; \ k = 1, \ldots, T - 1; \ s_k \in S_k) \\
& \quad \lambda_T^{(i)} \leq \frac{1}{I}, \quad (i = 1, \ldots, I) \\
& \quad \lambda_0 \geq 0 \\
& \quad \lambda_t^{(i)} \geq 0, \quad (t = 1, \ldots, T; \ i = 1, \ldots, I) \\
& \quad \omega \geq 0
\end{align*}
\]

where

\[
\begin{align*}
\eta_{j,k,k+1}^{(i)} &= r_{j,k+1}^{(i)} - \left( 1 + r_{k}^{(i)} \right) r_{jk}^{(i)}, \quad (k = 0, \ldots, T - 1) \\
\eta_{jkt}^{(i)} &= \left( 1 + r_{t-1}^{(i)} \right) \eta_{j,k,t-1}^{(i)}, \quad (k = 0, \ldots, T - 2; \ t = k + 2, \ldots, T) \\
\eta_{jkT}^{sk} &= \frac{1}{I} \sum_{i_k \in V^{sk}_k} \eta_{jkt}^{(i_k)}, \quad (k = 1, \ldots, T - 1) \\
F_1^{(i)} &= \left( 1 + r_0 \right) W_0; \ F_t^{(i)} = \left( 1 + r_{t-1}^{(i)} \right) F_{t-1}^{(i)}, \quad (t = 2, \ldots, T) \\
F_T &= \frac{1}{I} \sum_{i=1}^{I} F_T^{(i)}
\end{align*}
\]

Dual variables are:

Equation (21) : \(z_{j0}^{*}\), \( (j = 1, \ldots, n) \)

Equation (22) : \(z_{jk}^{sk}\), \( (j = 1, \ldots, n; \ k = 1, \ldots, T - 1; \ s_k \in S_k) \)

Equation (23) : \(q^{(i)}\), \( (i = 1, \ldots, I) \)

In the dual compact representation, we have \(TI + 2\) decision variables, and \(nM + I\) constraints, where \(M\) denotes the number of nodes. If a piece-wise linear risk measure, such as the first-order lower partial moment, is used, the number of the general constraints depends only on the number of decision nodes and risky assets, because the number of boundary constraints is equal to the number of sample paths. Therefore, we have only \(nM\) general constraints. The problem size associated with the computation time is drastically reduced \(^{12}\). We expect computation

\(^{11}\)The primal form is bothersome, but easy to derive.

\(^{12}\)In general, linear programming problem is formulated as follows;

\[
\begin{align*}
\text{Minimize} & \quad c^{T} x \\
\text{subject to} & \quad \bar{b} \leq Ax \leq \bar{b} \\
& \quad l \leq x \leq u
\end{align*}
\]

We have two kinds of constraints in this problem; general constraints (as in Equation 25)) and boundary constraints (Equation (26)). The bounded variables are specially treated in the algorithm and the mathematical programming
We examine numerical examples in order to compare the computation time of the dual compact formulation with that of the original formulation. Three periods and four assets problems are solved for four kinds of branching trees and six kinds of numbers of simulated paths.

Figure 13 shows original/dual ratios — computation time of the original formulation to computation time of the dual compact formulation. When the interior point method is used, the dual compact formulation can be solved about three times as fast as the original formulation as in the left side of Figure 13. When the dual form is solved using the simplex method, the computation time can be improved drastically as in the right side of Figure 13.

![Figure 13: Computation time: original/dual ratio](image)

### 7 Concluding Remarks

The hybrid optimization model using simulated paths and the decision tree allows both the describing of uncertainties with high accuracy and the making of conditional decisions. This paper has developed some techniques of generating the extended decision tree to make the conditional decisions in the simulated path framework, and then compare them. The findings in this paper show that the S-PBC method is better than other methods, such as the SQC method and the W-PBC method, but its parameter is expected to be dependent on the problem.

In this chapter, some cases have been tested using numerical examples, the results of which show some features of the hybrid model. The number of branching trees and the number of paths affect the efficient frontier of wealth and optimal solutions, but the appropriate relationship between the number of branching tree and the number of paths remains open to discussion and should be investigated by additional numerical tests. It has been shown that any investment strategy can be implemented according to the investment unit function, but fixed-proportion strategy seems to dominate other two strategies in this example (fixed-unit and fixed-value).

This model used here has been developed for asset allocation, but its concept can be widely applied to general financial problems, such as optimal ALM problem, optimal bond portfolio selection, and so on.

### Bibliography


**Bogentoft, E., Romeijn, H., and Uryasev, S.,** 2001, “Asset/Liability Management for software, and therefore the boundary constraints are treated differently from the general constraints. Detail is referred to the textbook of the linear programming (Dantzig and Thapa, 1997).


