Abstract

We need to solve a multi-period optimization problem to decide dynamic investment policies under various practical constraints. Hibiki (2001,2003,2006) develop the hybrid model where the conditional decision can be made in the simulation approach, and the investment proportions are expressed by the step function of the amount of wealth. In this paper, we introduce the idea of the state-dependent function into the hybrid model as well as Takaya and Hibiki (2012). At first, we define the state-dependent function form for the multiple asset allocation problem with CVaR (Conditional Value at Risk) using the hybrid model, and we clarify that the function form is V-shaped and kinked at the level of the VaR. We propose the piecewise linear model with the V-shaped function to solve the multi-period and state-dependent asset allocation problem. We solve the three period problem for five assets, and compare the piecewise linear model with the hybrid model. We conduct the sensitivity analysis for the different risk averse coefficients and autocorrelations to examine the characteristics of the model.

Keywords: finance, stochastic optimization, multi-period asset allocation, simulation

1. Introduction

Institutional investors need to manage their investment funds in consideration of rebalancing assets during the planning period for the efficient asset management. Individual investors who face their long-term financial planning have similar problems. The multi-period optimization model involving the dynamic investment decision explicitly can be used to solve the problem with several constraints in practice. There are many studies in the literatures of different academic fields. In the fields of mathematical finance and financial economics, the analytical solutions or approximate solutions are derived employing the HJB equation or Martingale method under the setting of the continuous time and continuous distribution. Cvitanić and Karatzas [3] derive the closed form solution of minimizing the first-order lower partial moment (LPM). Recently, several Monte Carlo methods have been developed for the computation of optimal portfolio policies (Detemple, Garcia and Rindisbacher [4]). But the number of assets is limited to solve the problem in general, and it is also difficult to derive the optimal solutions for the model with the practical constraints. In the fields of financial engineering, the numerical solutions are derived employing mathematical programming under the setting of the discrete time and discrete distribution in order to solve the multiple assets problem with the practical constraints. Hibiki [5] developed the simulated path model to solve the multi-period portfolio optimization problem. The return distribution is described using the simulated paths generated by the Monte Carlo.
Hibiki [6, 8, 9] developed the hybrid model involving the conditional decisions in the simulated path approach. In the hybrid model, similar states are bundled at each time so that the same conditional decisions can be made, though the decisions are not strictly state-dependent.

The solution methods of these two approaches are different from each other. When the optimization problem is solved analytically, the analytical solution is expressed as the state-dependent function. When the mathematical programming problem is solved numerically, the optimal solution is derived for the discrete decision variables. There does not exist the integrated model of these approaches except Takaya and Hibiki [13] which propose the linear approximation model to derive the state-dependent asset allocation in a discrete time and a discrete distribution and solve the two-asset problem with a risky asset and a riskless asset to compare the analytical solutions.

In this paper, we also introduce the idea of the state-dependent function into the hybrid model as well as Takaya and Hibiki [13], and construct the integrated model of both stochastic programming approach and analytical approach for multiple assets problem. Specifically, we propose the multi-period optimization model involving the state-dependent decision making with the conditional value at risk (CVaR) [10], rather than the lower partial moment as a risk measure. CVaR is the well-known risk measure defined as the expected loss based on a given loss tolerance.

The contributions and characteristics of our paper involving the comparison with the previous studies are in what follows.

1. **Definition of the state-dependent function for the CVaR**
   Using the hybrid model, we define the state-dependent function form for the problem where the objective function is the expected terminal wealth minus the CVaR of the terminal wealth. We clarify that the function form is V-shaped and kinked at the level of the VaR in the case of the CVaR risk measure. However, our conclusion is numerically drawn, but cannot be proved theoretically. We conduct the analysis for the three-asset problem including two risky assets and a riskless asset where two risky assets are correlated and autocorrelated with each other, and draw the conclusion. Moreover, we discuss the characteristics of the piecewise linear model through the analysis for five-assets problem.

2. **Overcoming the drawback of the hybrid model with respect to the optimal weights**
   The optimal weights are assumed to be expressed as a step function of the amount of wealth in the hybrid model because they are derived discretely for the decision nodes. Therefore, there is the drawback that the optimal weights may change significantly even if the amount of wealth is changed only slightly because the states belonging in the decision nodes lying next to each other may have similar amounts of wealth from each other. We can overcome the drawback by employing the state-dependent and continuous function.

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2The scenario tree model is widely used in the mathematical programming approach (Ziemba and Mulvey [16], Zenios and Ziemba [14, 15]). However, the numbers of decision variables and constraints in the scenario tree may grow exponentially to ensure that the constructed representative set of scenarios covers the set of possibilities to a sufficient degree.

3Takano and Gotoh [12] solve the optimization model with the CVaR in the simulated path approach as well as our paper, and a nonlinear control policy is derived using Kernel method. However, it is difficult to advice the investment action in practice because the weights derived discretely cannot be expressed explicitly by the state-dependent function. In this paper, it is possible to express the investment weights by the state-dependent function with respect to the amount of wealth in a real world.
(3) Consistent results with other risk measures

In the minimization problem of the first-order lower partial moments of wealth, Cvitanić and Karatzas [3] show that the optimal weight of a risky asset is zero if the amount of wealth is above the discounted value of the target wealth, and otherwise the optimal weight is expressed by the nonlinear function of wealth. In the same manner, Siegmann [11] shows that the optimal investment policy is V-shaped in terms of wealth in the case that the objective function is the expected wealth minus the second-order lower partial moment multiplied by the risk aversion. The optimal policy is V-shaped and state-dependent in terms of wealth, and kinked at the target level or threshold associated with the risk measure. The function form derived using the linear approximation model of our paper is consistent with that of the previous models.

(4) Developing the piecewise linear model with the state-dependent function for the more than three-asset problem

We develop the piecewise linear model for the N-asset problem involving the state-dependent investment unit function, and compare it with the hybrid model. We conduct the numerical analysis for the five-asset optimization problem and examine the usefulness of the model.

### Table 1: Comparison of our paper with the previous studies

<table>
<thead>
<tr>
<th>Authors</th>
<th>Year</th>
<th>No. of assets</th>
<th>Risk measure</th>
<th>Objective function</th>
<th>State variable</th>
<th>Function form</th>
<th>SF</th>
<th>SA</th>
<th>Model/Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cvitanić &amp; Karatzas[3]</td>
<td>1999</td>
<td>2</td>
<td>LPM(1) risk</td>
<td>min. wealth</td>
<td>nonlinear risk</td>
<td>Risk</td>
<td>No</td>
<td>No</td>
<td>Closed form solution</td>
</tr>
<tr>
<td>Hibiki[5]</td>
<td>2001</td>
<td>N</td>
<td>LPM(1) min.</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Simulated path model</td>
</tr>
<tr>
<td>Hibiki[6]</td>
<td>2001</td>
<td>N</td>
<td>LPM(1) min.</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Hybrid model</td>
</tr>
<tr>
<td>Hibiki[7]</td>
<td>2002</td>
<td>N</td>
<td>LPM(1) min.</td>
<td>past return</td>
<td>linear</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Simulated path model</td>
</tr>
<tr>
<td>Hibiki[9]</td>
<td>2006</td>
<td>N</td>
<td>LPM(1) min.</td>
<td>past return</td>
<td>linear</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Hybrid model</td>
</tr>
<tr>
<td>Calafiore [2]</td>
<td>2008</td>
<td>N</td>
<td>Variance risk</td>
<td>past return</td>
<td>linear</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>Convex programming</td>
</tr>
<tr>
<td>Takano &amp; Gotoh[12]</td>
<td>2011</td>
<td>N</td>
<td>CVaR weighted sum max.</td>
<td>past return</td>
<td>nonlinear</td>
<td>Yes</td>
<td>Yes</td>
<td>Model with Kernel function</td>
<td></td>
</tr>
<tr>
<td>Hirano &amp; Hibiki</td>
<td>2014</td>
<td>N</td>
<td>CVaR weighted sum max.</td>
<td>wealth</td>
<td>piecewise linear</td>
<td>Risk</td>
<td>Yes</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

†1 ‘N denotes ‘N ≥ 3’  
†2 ‘min. w. const.’ denotes ‘minimization with constraints’  
†3 If the weight function is the state-dependent function, then ‘Yes’.

‘Risk’ denotes SF with respect to risk measure  
†4 If the modeling is constructed in simulated path approach, then ‘Yes’.
We show Table 1 to compare our paper with previous studies of the simulated path approach and their referred studies, and clarify the differences.

This paper is organized as follows. In Section 2, we clarify the relationship between wealth and investment weights using the hybrid model. In Section 3, we propose the optimization model with a piecewise linear function which is state-dependent in terms of wealth for the CVaR problem. In Section 4, we solve the five-asset and three-period problems. The piecewise linear model is compared with the hybrid model, and the usefulness of the model is examined. In addition, we conduct the sensitivity analysis for two kinds of parameters; risk aversion and autocorrelation. Section 5 provides our concluding remarks.

2. Modeling using state-dependent function with CVaR

2.1. Hybrid model and conditional decision

The hybrid model [6, 8, 9] allows conditional decisions to be made for the similar states bundled at each time using the sample returns generated by the Monte Carlo method. We can use tree or lattice structure to make the conditional decisions. The rule that the same investment decisions are made in similar states is defined to satisfy the non-anticipativity.

![Figure 1: Hybrid model structures](image)

We employ the lattice structure as the modeling structure with respect to the decision nodes in this paper. We call ‘hybrid Nm model’ for the hybrid model with m decision nodes at each time hereafter. A decision node is defined as a set of states at each time, and a path is defined as a set of states through time. As examples of the lattice structure, we depict the hybrid N1 model in the left-hand side of Figure 1 and the hybrid N4 model in the right-hand side, respectively. The number of paths is twelve and the same decision is made for paths in each node at each time.

Hibiki [7] proposes the investment unit function to express the decision rule which is defined to satisfy the non-anticipativity condition in the simulated path model. Hibiki [8, 9]

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4 The simulated path approach is proposed by Hibiki [5, 6] where sample paths are generated using the Monte Carlo method and the optimal solutions are derived in the multi-period optimization model. The simulated path model [5] and the hybrid model [6] are the basic models.

5 Bogentoft, Romeijn and Uryasev [1] solve the ALM problem for pension funds using a hybrid model with a lattice structure by reference to Hibiki [6].

6 The simulated path model [5] corresponds to the hybrid N1 model (with a decision node at each time).
also proposes the investment unit function in the hybrid model to describe the decision rule in the similar states. The investment unit function is defined as Equation (2.1) and we describe the path-dependent investment unit for path $i$, time $t$ and decision node $s$ (decision variable $z_{jt}^s$),

$$ h^{(i)}(z_{jt}^s) = a^{(i)}_{jt} z_{jt}^s $$

(2.1)

where $a^{(i)}_{jt}$ is the investment unit parameter which depends on the decision rule or the investment strategy. There are various ways to decide the parameter values of $a^{(i)}_{jt}$, and we show two investment strategies as follows.\(^7\)

1. **Fixed-unit strategy:** $a^{(i)}_{jt} = 1$, or $z_{jt}^s$ denotes the investment unit

2. **Fixed-proportion strategy:** $a^{(i)}_{jt} = W^{(i)}_{tjt}$, or $z_{jt}^s$ denotes the investment proportion, where

   $\rho^{(i)}_{jt}$ is the price of risky asset $j$ of path $i$ at time $t$ and $W^{(i)}_{tjt}$ is the amount of wealth of path $i$ at time $t$

We can simply describe the hybrid models involving various decision rules by employing the investment unit functions.

In this paper, we use the hybrid model involving the lattice structure to decide the investment with the state-dependent step function. When we have $I$ states and $m$ nodes at each time, we have (about) $\frac{I}{m}$ states in each node. We sort the states by the amount of wealth at each time.

### 2.2. Formulation of the hybrid model

We formulate the hybrid model for the asset allocation problem. The terminal loss with respect to the amount of wealth is defined as the deviation from the amount of initial wealth, and both VaR and CVaR of the loss distribution are calculated.

The objective function is expressed by the expected terminal wealth minus CVaR multiplied by the constant $\gamma$. The notations used in the formulation are as below.

**1) Notations**

1. **Sets**
   - $S_t$: set of fixed-decision nodes $s_t$ at time $t$, ($t = 1, \ldots, T - 1$)
   - $V_t^*$: set of paths passing any node $s_t$ at time $t$, ($t = 1, \ldots, T - 1$; $s_t \in S_t$)

2. **Parameters**
   - $n$: number of risky assets
   - $T$: number of periods
   - $I$: number of sample paths
   - $\rho_{j0}$: price of risky asset $j$ at time 0, ($j = 1, \ldots, n$)
   - $\rho^{(i)}_{jt}$: price of risky asset $j$ of path $i$ at time $t$, ($j = 1, \ldots, n$; $t = 1, \ldots, T$; $i = 1, \ldots, I$)
   - $r_0$: interest rate in period 1
   - $r^{(i)}_{t-1}$: interest rate in period $t$ (the rate of path $i$ at time $t - 1$ is used), ($t = 2, \ldots, T$; $i = 1, \ldots, I$)
   - $W_0$: initial amount of wealth.
   - $\gamma$: risk aversion

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\(^7\)Hibiki [7] formulates the models with the investment unit functions which show the trend following strategy, the contrarian strategy, and the strategy using the autocorrelation. The investment unit function is expressed as a linear function of the previous rate of return.
Variables

\( z_{j0} \): investment unit for asset \( j \) and time 0, \((j = 1, \ldots, n)\)

\( z_{jtt}^{st} \): base investment variable for asset \( j \), time \( t \), and node \( s_t \)
\((j = 1, \ldots, n; t = 1, \ldots, T - 1; s_t \in S_t)\)

\( W_t^{(i)} \): amount of wealth of path \( i \) at time \( t \), \((t = 1, \ldots, T; i = 1, \ldots, I)\)

\( \text{VaR}(\beta) \): VaR at a \( \beta \) confidence level

\( \text{CVaR}(\beta) \): CVaR at a \( \beta \) confidence level

\( q^{(i)} \): deviation of the loss above VaR, \((i = 1, \ldots, I)\)

(2) Formulation

The hybrid \( Nm \) model is formulated as follows.

Maximize
\[
\frac{1}{I} \sum_{i=1}^{I} W_T^{(i)} - \gamma \cdot \text{CVaR}(\beta)
\]  
(2.2)

subject to

\[
\sum_{j=1}^{n} \rho_{j0} z_{j0} = W_0
\]  
(2.3)

\[
W_1^{(i)} = \sum_{j=1}^{n} \rho_{j1} z_{j0} \quad (i = 1, \ldots, I)
\]  
(2.4)

\[
W_t^{(i)} = \sum_{j=1}^{n} \left\{ \rho_{jt}^{(i)} - \left(1 + r_{t-1}^{(i)}\right) \rho_{jt-1}^{(i)} \right\} h^{(i)}(z_{jt-1}^{s_{t-1}}) + \left(1 + r_{t-1}^{(i)}\right) W_{t-1}^{(i)} \\
(t = 2, \ldots, T; s_{t-1} \in S_{t-1}; i \in V_t^{s_{t-1}})
\]  
(2.5)

\[
W_T^{(i)} + \text{VaR}(\beta) + q^{(i)} \geq W_0 \quad (i = 1, \ldots, I)
\]  
(2.6)

\[
\text{CVaR}(\beta) = \text{VaR}(\beta) + \frac{1}{(1 - \beta)I} \sum_{i=1}^{I} q^{(i)}
\]  
(2.7)

\[
h^{(i)}(z_{jt-1}^{s_{t-1}}) = \left( \frac{W_t^{(i)} {t-1}}{\rho_{jt-1}^{(i)}} \right) z_{jt-1}^{s_{t-1}} \quad (j = 1, \ldots, n; t = 2, \ldots, T; s_{t-1} \in S_{t-1}; i \in V_t^{s_{t-1}})
\]  
(2.8)

\[
z_{j0} \geq 0 \quad (j = 1, \ldots, n)
\]  
(2.9)

\[
z_{jtt}^{st} \geq 0 \quad (j = 1, \ldots, n; t = 1, \ldots, T - 1; s_t \in S_t)
\]  
(2.10)

\[
q^{(i)} \geq 0 \quad (i = 1, \ldots, I)
\]  
(2.11)

Equation (2.3) is the budget constraint which shows the full investment in risky assets. Equation (2.5) is the calculation of the amount of wealth at time \( t \) derived by the investment at time \( t - 1 \). Equations (2.6) and (2.7) are used to calculate CVaR(\( \beta \)).

The optimal weights are derived numerically for the discrete decision nodes including a set of states in the hybrid \( Nm \) model with fixed-proportion strategy, and therefore they are not strictly state-dependent.\(^8\) We examine the relationship between the amount of wealth and the investment weights using the hybrid N25 model in order to introduce the idea of

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\(^8\)The optimal weight is supposed to be a step function of the amount of wealth implicitly. It is not constrained that the optimal weight is a step function explicitly in the formulation, but it is expected because the decision nodes are generated discretely by employing the sorted amounts of wealth.
the state-dependent function derived in the analytical approach into the hybrid model in the next section.

2.3. Numerical analysis using the hybrid model

The relationship is examined at each time using the hybrid model with three assets, three periods and 25 nodes in order to approximate the state-dependent function. Details of the numerical analysis are as follows.

- Three assets consist of two risky assets and a riskless asset.
  - The rates of return of risky assets are normally distributed. The relationship is investigated using the parameters of three kinds of correlations between two risky assets \((\rho = -0.5, 0, 0.5)\) and three kinds of autocorrelations \((c = -0.5, 0, 0.5)\).
  - The following correlation matrices are supposed, where \(A_i(t)\) denotes asset \(i\) at time \(t\).

<table>
<thead>
<tr>
<th>Correlation</th>
<th>Cross-correlation (lag 1)</th>
<th>Cross-correlation (lag 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_1(t))</td>
<td>(A_1(t + 1)) (c) (\rho c)</td>
<td>(A_1(t + 2)) (0.5c) (0.5\rho c)</td>
</tr>
<tr>
<td>(\rho)</td>
<td>(\rho c) (c)</td>
<td>(0.5\rho c) (0.5c)</td>
</tr>
</tbody>
</table>

- Both the expected value and standard deviation of the rate of return of risky asset 1 are larger than those of risky asset 2. \((\bar{r}_1 = 5\%, \sigma_1 = 20\%; \bar{r}_2 = 2\%, \sigma_2 = 5\%)\)
- Risk-free rate is 0%.
- Initial wealth is 100 million yen.
- The CVaRs are calculated at 80% confidence level with 50,000 sample paths.
- The problem is solved using the iterative algorithm developed by Hibiki [9]. We utilize the hybrid N1 model with the fixed-unit strategy in the first step, and the fixed-proportion strategy in the second step. We utilize the hybrid N25 model with the fixed-proportion strategy in the subsequent steps until the objective function converges. In the hybrid Nm model, the lattice structure is generated as in Figure 1 by sorting the amounts of wealth calculated at each time after each iteration, and dividing them into \(m\) sets. The number of states in a decision node is 2,000 in the analysis because of 50,000 paths and 25 nodes.
- Two cases are analyzed as follows.
  
  (Case 1) Optimization problems are solved for three kinds of risk-averse coefficients under the conditions that (1a) no cash borrowing is allowed, and (1b) cash borrowing which limit ratio is 100% is allowed.

  (Case 2) CVaR minimization problems are solved for the combination of three kinds of correlations and three kinds of autocorrelations under the condition that cash borrowing is allowed.

2.3.1. Case 1a

We show the scatter plot of the amounts of wealth vs. the investment weights to examine the relationship between them for no correlation and no autocorrelation case \((\rho = 0, c = 0)\) in Figure 2. We can find the relationship in Figure 2, and the optimal weights are almost V-shaped with respect to the amount of wealth.\(^9\) Specifically, the weights of risky assets are the smallest at the VaR level of wealth, \(W_0 - \text{VaR}(\beta)\), which is the initial wealth minus the value at risk of the terminal wealth. The optimal weights tend to increase as the amount

\(^9\)The optimal weights may not look V-shaped for the small or large amounts of wealth. However, this reason is that we have the same weights in the first node for the small amounts of wealth, and in the 25th node for the large amounts of wealth, respectively.
of wealth moves away from the VaR level. These results are consistent with Cvitanić and Karatzas [3] and Siegmann [11] which show that the optimal investment policies are kinked at the target wealth, and piecewise or V-shaped in terms of wealth under the lower partial moment (LPM). The VaR corresponds to the target wealth in the case of calculating the CVaR. The target levels are different between the CVaR and the LPM, however they are used in calculating downside risk measures. Therefore, it is reasonable the optimal weights are kinked at the VaR when calculating the CVaR.

We have the different point between the CVaR and the LPM with respect to the function form for the risk minimization problem. When the problem is solved by minimizing the LPM, the optimal weights of risky assets are zero for the amount of wealth above the discounted target wealth. However, the optimal weights of risky assets for CVaR minimization problem are not zero even above the target ($W_0 - \text{VaR}(\beta)$). This reason is that the terminal wealth and its VaR can be controlled and the CVaR can be improved by investing the risky assets during the planning period. This is the different point from the LPM where the target is constant when the CVaR is used as the risk measure.

The weight function of asset 1 with larger expected return and volatility is clearly V-shaped in terms of wealth. On the other hand, the weight of asset 2 is also V-shaped around

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10The objective function of Cvitanić and Karatzas [3] is minimization of the first-order LPM, and the amount of wealth corresponding to the kink of the piecewise function is the discounted value of the target wealth. On the other hand, the objective function of Siegmann [11] is the expected asset values minus the second-order LPM multiplied by the level of risk aversion, and the assets of the kink of the V-shaped function is the target asset minus a certain value. It is expected that the location of the kink in terms of the amount of wealth is a discounted value of $W_0 - \text{VaR}(\beta)$ when the CVaR is minimized. Though the modification of $W_0 - \text{VaR}(\beta)$ should be applied strictly, it is difficult to find the location of the kink for the objective function which is the expected wealth minus the CVaR multiplied by the risk aversion. Therefore, we employ $W_0 - \text{VaR}(\beta)$ as the location of kink in terms of wealth. The method of modification is our future research.
the VaR level, but it does not keep its shape in larger and smaller amounts of wealth because of the non-negative constraint of cash. Next, we solve the problem without the non-negative constraint of cash to examine the reason.

2.3.2. Case 1b

The weights of risky assets are subject to the cash borrowing constraint because the upper bound of the sum of the weights is one in the Case 1a where no cash borrowing is allowed. We compare two cases between the cash borrowing allowed and no cash borrowing to examine the effect of the cash borrowing constraint. We show the results for $\gamma = 4$ in Figure 3. The results of no cash borrowing case are the same as Figure 2.

![Figure 3: Comparison between two cash borrowing cases (Case 1b)](image)

We notice that the two kinds of functions are not overlapped even if the sum of the weights are below one because the value of $W_0 - \text{VaR}(0.8)^+$ in the cash borrowing case are larger than the value in the no cash borrowing case. We find that the weight of asset 2 can be also V-shaped in larger and smaller amounts of wealth with cash borrowing allowed. This shows that the weights are subject to the upper bound for the small or large amounts of wealth in Case 1a.

2.3.3. Case 2

We also examine the relationship between wealth and weights for the combination of three kinds of correlations and three kinds of autocorrelations. We show the results at time 1 in Figure 4.

We can also suppose that the state-dependent functions are V-shaped and kinked at the VaR as well as in Case 1.
2.3.4. Sensitivity of the weight to the amount of wealth

As previous figures are examined, we find the slopes of the function are different between above and below \( W_0 - \text{VaR}(\beta) \). The slopes of the function show the sensitivities of the weight to the amount of wealth. We examine the difference between above and below \( W_0 - \text{VaR}(\beta) \).

The sensitivity is expressed as

\[
\text{Sensitivity} = \frac{\text{Change in weight(\%)} \text{ from the weight of '} W_0 - \text{VaR}(\beta) \text{'} \text{}}{\text{Change in wealth(million yen) from '} W_0 - \text{VaR}(\beta) \text{'} \text{}}.
\]

We can derive the sensitivity or the slope of the function using regression. We show the sensitivity of the weight to the amount of wealth in Table 2 for the different risk aversive coefficients of Case 1a. The absolute sensitivity below \( W_0 - \text{VaR}(\beta) \) is larger than the sensitivity above \( W_0 - \text{VaR}(\beta) \) because we need to increase the amount of wealth by taking risk or increasing the weights of risky assets. The absolute sensitivity becomes small to avoid taking risk regardless whether the amount of wealth is above or below \( W_0 - \text{VaR}(\beta) \) as the risk aversive coefficient(\( \gamma \)) becomes large.

2.3.5. Findings

These results show the two useful findings below.

- It is useful to employ the hybrid model in order to find the state-dependent function form in the simulated path approach.
- The state-dependent function form is affected by the critical value to evaluate the risk measure such as the target wealth for the LPM and the VaR in terms of the amount of wealth for the CVaR.
Table 2: Sensitivity of the weight to the amount of wealth

<table>
<thead>
<tr>
<th>Asset 1</th>
<th>( \gamma = )</th>
<th>( \gamma = 2 )</th>
<th>( \gamma = 3 )</th>
<th>( \gamma = 4 )</th>
<th>( \gamma = \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Above ( W_0 - \text{VaR}(0.8) )</td>
<td>time 1</td>
<td>1.79</td>
<td>1.46</td>
<td>1.33</td>
<td>0.78</td>
</tr>
<tr>
<td></td>
<td>time 2</td>
<td>1.63</td>
<td>1.46</td>
<td>1.33</td>
<td>0.82</td>
</tr>
<tr>
<td>Below ( W_0 - \text{VaR}(0.8) )</td>
<td>time 1</td>
<td>-2.64</td>
<td>-2.53</td>
<td>-2.47</td>
<td>-2.33</td>
</tr>
<tr>
<td></td>
<td>time 2</td>
<td>-2.67</td>
<td>-2.62</td>
<td>-2.47</td>
<td>-2.39</td>
</tr>
<tr>
<td>Asset 2</td>
<td>( \gamma = )</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>( \gamma = \infty )</td>
</tr>
<tr>
<td>Above ( W_0 - \text{VaR}(0.8) )</td>
<td>time 1</td>
<td>11.47</td>
<td>9.12</td>
<td>8.79</td>
<td>5.57</td>
</tr>
<tr>
<td></td>
<td>time 2</td>
<td>8.35</td>
<td>7.72</td>
<td>7.67</td>
<td>5.10</td>
</tr>
<tr>
<td>Below ( W_0 - \text{VaR}(0.8) )</td>
<td>time 1</td>
<td>-14.21</td>
<td>-14.47</td>
<td>-14.76</td>
<td>-13.99</td>
</tr>
<tr>
<td></td>
<td>time 2</td>
<td>-16.40</td>
<td>-16.11</td>
<td>-16.20</td>
<td>-15.98</td>
</tr>
</tbody>
</table>

3. Piecewise Linear Model

3.1. Developing the investment unit function with CVaR

We adopt the piecewise linear function form to introduce the state-dependent function (i) because the weights are nearly proportional to the amount of wealth in Figures 2 to 4, and (ii) so that the large scale problem can be solved in the simulated path approach. However, the sum of the weights for risky assets needs to be subject to the cash constraints in largest and smallest amounts of wealth and the non-negativity constraints. Therefore, we propose the investment unit function in Equation (3.1) with the investment weight \( w_{jt}(i) \) which is a piecewise linear function of the amount of wealth at each time \( W_t(i) \) in Equation (3.2). We call it the piecewise linear model.

\[
h^{(i)}(w_{jt}(i)) = \begin{cases} 
\frac{W_t(i)}{\rho_t(i)} w_{jt} (j = 1, \ldots, n; t = 1, \ldots, T - 1; i = 1, \ldots, I), \\
\end{cases}
\]

\[
w_{jt}(i) = \begin{cases} 
a_{jt}^1 + b_{jt}^1 \theta_{jt}^1 & (W_t(i) \leq \theta_{jt}^1) \\
a_{jt}^1 + b_{jt}^1 W_t(i) & (\theta_{jt}^1 < W_t(i) \leq W_0 - \text{VaR}(\beta)) \\
a_{jt}^2 + b_{jt}^2 W_t(i) & (W_0 - \text{VaR}(\beta) < W_t(i) \leq \theta_{jt}^2) \\
a_{jt}^2 + b_{jt}^2 \theta_{jt}^2 & (W_t(i) > \theta_{jt}^2) \\
\end{cases} 
\]

\[
v_{jt} = \begin{cases} 
a_{jt}^1 + b_{jt}^1 (W_0 - \text{VaR}(\beta)) & (j = 1, \ldots, n; t = 1, \ldots, T - 1; i = 1, \ldots, I) \\
a_{jt}^2 + b_{jt}^2 (W_0 - \text{VaR}(\beta)) & (j = 1, \ldots, n; t = 1, \ldots, T - 1) \\
\end{cases}
\]

where \( a_{jt}^1 \) and \( a_{jt}^2 \) are the intercept coefficients, \( b_{jt}^1 \) and \( b_{jt}^2 \) are the slope coefficients, \( \theta_{jt}^1 \) is the boundary of the amount of wealth where the weight of risky asset \( j \) becomes constant below \( W_0 - \text{VaR}(\beta) \), and \( \theta_{jt}^2 \) is the boundary of the amount of wealth above \( W_0 - \text{VaR}(\beta) \). Specifically, \( \theta_{jt}^1 \) and \( \theta_{jt}^2 \) are determined, subject to \( \sum_{j=1}^{n} w_{jt}(i) \leq 1 - L_C \) and \( w_{jt}(i) \geq 0 \), where \( L_C \) is the lower bound of cash. The formulation can be depicted as in Figure 5.

Equation (3.3) is the constraint which imposes the condition that the weights of two linear functions coincide with each other at the wealth level of \( W_0 - \text{VaR}(\beta) \). We need to utilize the iterative algorithm to solve the problem as well as the hybrid model. Details
are explained in Section 3.3. We show the examples of the relationship between wealth and investment weights for the piecewise linear model and the hybrid model on the left-hand side of Figure 6.

The state-dependent function of the proposed model is the continuous V-shaped function form, while that of the hybrid model is the discrete step function form. The investment decision for the hybrid model with a small number of nodes may be dependent on the method of separating the decision nodes. On the other hand, the piecewise linear model with the V-shaped function is expected to have the same structure with small number of decision variables as the hybrid model with a lot of decision nodes which needs a lot of decision variables. The both discrete and continuous functions are expected to be similar to each other as on the left of Figure 6. In addition, the investment proportions may change drastically in the hybrid model even if the states belonging in the decision nodes lying next to each other do not have different amounts of wealth. This can be resolved by using the piecewise linear model.

Next, we show the example of the piecewise linear functions for three risky assets on the left of Figure 6. The dotted lines show the functions with short sales for risky assets and cash borrowing allowed, and the solid lines with data markers show the functions without short sales and cash borrowing allowed.11

---

11Strictly, the dotted lines are expected to be located slightly on the right side of the solid lines as in
The piecewise linear model is one of the variations of the hybrid model with two decision nodes where the partition boundary of two nodes is $W_0 - \text{VaR}(\beta)$. We can apply the formulation of the hybrid model in Section 2.2 into the piecewise linear model with some modifications.

### 3.2. Formulation

We formulate the piecewise linear model based on the hybrid model. We also use the sets, parameters and the variables defined in Section 2.2. In this section, we describe the notations of the alternative decision variables.\(^{(1)}\)

**Alternative decision variables**

1. **$w_{jt}$**: [Unit (N1)] investment unit for asset $j$ and time $t$ ($j = 1, \ldots, n; t = 1, \ldots, T - 1$)
2. **$w_{jt}^{(i)}$**: [PwL] investment proportion for asset $j$, time $t$ and path $i$ ($j = 1, \ldots, n; t = 1, \ldots, T - 1; i = 1, \ldots, I$)
3. **$a_{jt}^{u}, b_{jt}^{u}$**: [PwL] intercept and slope of the piecewise linear function of the investment proportion for asset $j$ and time $t$ ($u = 1, 2; j = 1, \ldots, n; t = 1, \ldots, T - 1$)

**Formulation**

We can formulate the piecewise linear model by modifying the formulation of the hybrid model in Section 2.2. The constraints (3.1) to (3.3) are added to the formulation. In addition, we replace the constraints (2.5) with the following constraints (3.4), and (2.10) with (3.5), respectively.

\[
W_t^{(i)} = \sum_{j=1}^{n} \left\{ \rho_j^{(i)} - \left( 1 + r_{t-1}^{(i)} \right) \rho_j^{(i)}_{t-1} \right\} b_{jt}^{(i)}(w_{jt}^{(i)} - 1 + r_{t-1}^{(i)}) W_{t-1}^{(i)}
\]
\[
(\text{for } t = 2, \ldots, T; i = 1, \ldots, I)
\]
\[
w_{jt}^{(i)} \geq 0, (j = 1, \ldots, n; t = 1, \ldots, T - 1; i = 1, \ldots, I),
\]

### 3.3. Iterative algorithm for the piecewise linear model

The weights of the risky assets are expressed by the state-dependent and piecewise linear function of the amount of wealth which is divided into four lines by the VaR and two boundaries with respect to the terminal wealth as in Equation (3.2). In addition, VaR($\beta$), $\theta_j^1$ and $\theta_j^2$ are not given as parameter values, but they need to be determined by solving the problem. It is difficult to solve the problem because (i) it is formulated with the non-linear and non-convex function, and (ii) the piecewise linear function is divided at VaR($\beta$) and two boundaries, determined by the amount of wealth. Therefore, we solve the problem using the modification of iterative algorithm developed by Hibiki [9].

The problem can be described as the linear programming problem by using the amounts of wealth, VaR, and two boundaries calculated in the previous iteration as the parameters, and therefore we can derive the optimal solutions by solving it iteratively. The algorithm has the following six steps.

---

\(^{(1)}\) The investment unit for asset $j$ and time 0 is also denoted by $z_j0$, as well.
Step 1: We set up \( h^{(i)}(w_{jt}) = w_{jt} \) as the investment unit function. We solve a problem using the hybrid N1 model with the fixed-unit strategy.\(^{13}\) Let \( Obj_0 \) denote the objective function value, and set \( k = 1 \).

Step 2: Let \( W_{t(k-1)}^{(i)*} \) be the amount of wealth of path \( i \) at time \( t \), and we calculate it. We set up \( h^{(i)}(w_{jt}) = \left( \frac{W_{t(k-1)}^{(i)*}}{\rho_{jt}^{(i)}} \right) w_{jt} \) as the investment unit function at the \( k \)-th iteration, and solve the problem using the hybrid N1 model with the fixed-proportion strategy. We calculate the objective function value \( Obj_k \), and set \( k \leftarrow k + 1 \).

Step 3: Go to Step 4 if a value \( Obj_k - Obj_{k-1} \) is lower than a tolerance. Otherwise, return to Step 2.

Step 4: \( W_{t(k-1)}^{(i)*} \) and \( \text{VaR}(\beta)_{k-1} \) are calculated as the amount of wealth of path \( i \) at time \( t \) and VaR of the \((k-1)\)th iteration, respectively. We solve the problem using Equations (3.8)-(3.10) instead of Equations (3.1)-(3.3), with short sales allowed and without cash constraint. Set \( k \leftarrow k + 1 \).

\[
\begin{align*}
    h^{(i)}(w_{jt}) &= \left( \frac{W_{t(k-1)}^{(i)*}}{\rho_{jt}^{(i)}} \right) w_{jt} \quad (t = 1, \ldots, T - 1), \quad (3.8) \\
    w_{jt}^{(i)} &= \left\{ \begin{array}{ll}
        a_{jt}^1 + b_{jt}^1 W_{t(k-1)}^{(i)*} & (W_{t(k-1)}^{(i)*} \leq W_0 - \text{VaR}(\beta)_{k-1}) \\
        a_{jt}^2 + b_{jt}^2 W_{t(k-1)}^{(i)*} & (W_{t(k-1)}^{(i)*} > W_0 - \text{VaR}(\beta)_{k-1})
    \end{array} \right., \quad (3.9) \\
    a_{jt}^1 + b_{jt}^1 (W_0 - \text{VaR}(\beta)_{k-1}) &= a_{jt}^2 + b_{jt}^2 (W_0 - \text{VaR}(\beta)_{k-1}). \quad (3.10)
\end{align*}
\]

Step 5: \( W_{t(k-1)}^{(i)*} \), \( \text{VaR}(\beta)_{k-1} \), \( \theta_{jt, t(k-1)}^{(1)*} \) and \( \theta_{jt, t(k-1)}^{(2)*} \) are determined as the boundaries of the amounts of wealth, using \( W_{t(k-1)}^{(i)*} \), \( \text{VaR}(\beta)_{k-1} \), \( \theta_{jt, t(k-1)}^{(1)*} \) and \( \theta_{jt, t(k-1)}^{(2)*} \) of the \((k-1)\)th iteration, respectively. Specifically, \( \theta_{jt, t(k-1)}^{(1)*} \) and \( \theta_{jt, t(k-1)}^{(2)*} \) are determined as

\[
\begin{align*}
    w_{jt}^{(i)} &= \left\{ \begin{array}{ll}
        a_{jt}^1 + b_{jt}^1 \theta_{jt, t(k-1)}^{(1)*} & (W_{t(k-1)}^{(i)*} \leq \theta_{jt, t(k-1)}^{(1)*}) \\
        a_{jt}^2 + b_{jt}^2 \theta_{jt, t(k-1)}^{(1)*} & (\theta_{jt, t(k-1)}^{(1)*} \leq W_{t(k-1)}^{(i)*} \leq W_0 - \text{VaR}(\beta)_{k-1}) \\
        a_{jt}^2 + b_{jt}^2 \theta_{jt, t(k-1)}^{(2)*} & (W_0 - \text{VaR}(\beta)_{k-1} < W_{t(k-1)}^{(i)*} \leq \theta_{jt, t(k-1)}^{(2)*}) \\
        a_{jt}^2 + b_{jt}^2 \theta_{jt, t(k-1)}^{(2)*} & (W_{t(k-1)}^{(i)*} > \theta_{jt, t(k-1)}^{(2)*})
    \end{array} \right., \quad (3.11)
\end{align*}
\]

We calculate the objective function value \( Obj_k \) using the optimal solutions.

Step 6: Stop if a value \( Obj_k - Obj_{k-1} \) is lower than a tolerance. Otherwise, set \( k \leftarrow k + 1 \), and return to Step 5.

The algorithm does not guarantee to derive the global optimal solutions. This algorithm is a heuristic one, and any solutions derived may be locally optimal.\(^{14}\)

\(^{13}\)We replace Equations (2.8) and (2.10) with Equation (3.6) for the hybrid N1 model with fixed-unit strategy, or Equation (3.7) for the hybrid N1 model with fixed-proportion strategy, respectively.

\[
\begin{align*}
    \text{[Unit (N1)]} & \quad h^{(i)}(z_{jt}) = z_{jt} \geq 0, \quad (j = 1, \ldots, n; \quad t = 1, \ldots, T - 1), \quad (3.6) \\
    \text{[Proportion (N1)]} & \quad h^{(i)}(z_{jt}) = \left( \frac{W_{t(k-1)}^{(i)*}}{\rho_{jt}^{(i)}} \right) z_{jt}; \quad z_{jt} \geq 0, \quad (j = 1, \ldots, n; \quad t = 1, \ldots, T - 1). \quad (3.7)
\end{align*}
\]

We also replace \( h^{(i)}(z_{jt}^{(j-1)}) \) with \( h^{(i)}(z_{jt}) \) in Equation (2.5).

\(^{14}\)Empirically, the objective function value almost converges after two iterations in Step 2 and five iterations in Step 5 in the numerical analysis after Section 4.2. Therefore we conduct the analysis fixing the
The weights are decision variables in the hybrid model, but these four kinds of coefficients are decision variables for each asset at time $t$ in the piecewise linear model. The number of decision variables of the hybrid $N_m$ model is $m$ for each asset at time $t$, and thus we can compare it with the hybrid N4 model under the same condition. Therefore, we need to pay attention to the fact that the objective function of the hybrid $N_m$ model for $m \neq 4$ is not comparable to that of the piecewise linear model to assess the advantages of the models.

4. Numerical Analysis

4.1. Setting

We solve the three-period model for five assets; domestic stock (DS), foreign stock (FS), domestic bond (DB), foreign bond (FB) and cash. Foreign currency risk is hedged completely for the foreign assets (FS and FB). Therefore, the rates of return of assets are estimated on a Japanese yen basis. A period is a month. The expected values and standard deviations of rates of asset return and the correlation coefficients among assets are estimated based on the indices from January 2004 to December 2013 as in Tables 3 and 4. Estimated correlation coefficients are the estimates with $c = 1$ in Table 4. We append the parameter $c$ because of the sensitivity analysis. We use Nikkei 225 index for the DS, S&P 500 for the FS, long-term Japanese government bond futures for the DB, and 10-year U.S. Treasury note futures for the FB.

<table>
<thead>
<tr>
<th>Table 3: Expected values and standard deviations of rates of return</th>
</tr>
</thead>
<tbody>
<tr>
<td>DS</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td>Expected value</td>
</tr>
<tr>
<td>Standard deviation</td>
</tr>
</tbody>
</table>

† The values for foreign assets are on a Japanese yen basis.

The interest rates are 0.0214% in the first period, 0.0269% in the second period, and 0.0317% in the third period. These rates are estimated, based on the 1,2,3 month Japanese yen LIBOR interest rates in the ten years. Initial amount of wealth is a hundred million yen, and the number of sample paths is 10,000. We calculate CVaR at a 80% confidence level. No short sales are allowed for risky assets, and the upper limit for cash is 10% of total assets. We generate sample paths of the rate of return of assets which are assumed to number of iterations instead of the convergence condition in Steps 3 and 6. The objective function value in Hibiki [9] converges after two iterations. The optimal solutions of the hybrid N4 model which is compared with the piecewise linear model can be derived from Steps 1 to 3. The optimal solutions of the hybrid N4 model can be derived using the following Step 4’ instead of the Steps 4 and 5 for the piecewise linear model.

Step 4’ : $W_{t(k-1)}^{(i)}$, is calculated as the amount of wealth of path $i$ at time $t$ of the $(k-1)$th iteration. We sort the amount of wealth $W_{t(k-1)}^{(i)}$ at each time and divide $m$ nodes to solve the hybrid $N_m$ model.

We solve the problem using the following investment unit function,

$$h_{(i)}^{(k)}(z_{jt}) = \left( \frac{W_{t(k-1)}^{(i)}}{p_{jt}^{(i)}} \right) z_{jt}$$

We calculate the objective function value $Obj_k$ using the optimal solutions.

The objective function value almost converges after three iterations in Step 4’.

15Strictly, the degree of freedom of the piecewise linear model is lower than that of the hybrid N4 model because of Equation (3.3). However, we do not discuss the degree of freedom of the models hereafter.
Table 4: Correlation matrices

<table>
<thead>
<tr>
<th>Correlation</th>
<th>First cross-correlation(lag 1)</th>
<th>Second cross-correlation(lag 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DS(t) FS(t) DB(t) FB(t)</td>
<td>DS(t+1) FS(t+1) DB(t+1) FB(t+1)</td>
</tr>
<tr>
<td>DS(t)</td>
<td>1.00 0.61 −0.38 −0.37</td>
<td>0.19c 0.23c −0.13c −0.28c</td>
</tr>
<tr>
<td>FS(t)</td>
<td>1.00 0.02 −0.09</td>
<td>0.18c 0.18c −0.17c −0.27c</td>
</tr>
<tr>
<td>DB(t)</td>
<td>1.00 0.74</td>
<td>0.05c 0.04c −0.01c 0.15c</td>
</tr>
<tr>
<td>FB(t)</td>
<td>1.00</td>
<td>0.05c 0.04c 0.02c 0.08c</td>
</tr>
</tbody>
</table>

be normally distributed, and calculate the asset prices. Specifically, we generate sample paths using the parameters of Tables 3 and 4.

All of the problems are solved using NUOPT (Ver. 13.1) – mathematical programming software package developed by NTT DATA Mathematical System, Inc. – on Windows 7 personal computer which has Core i5-2540M 2.6 GHz CPU and 8 GB memory.

4.2. Base case
We solve the problems with 0.6 risk aversion ($\gamma = 0.6$) and estimated autocorrelation ($c = 1$) using the piecewise linear model (called 'PwL model' hereafter) and the hybrid N1, N2 and N4 models, and compare these results. It is expected that the objective function of the hybrid N$m$ model is large as $m$ is large because the number of decision variables is almost proportion to the number of decision nodes. As mentioned before, we can compare the PwL model with the hybrid N4 model under the same condition, but it is expected that both the PwL function of the proposed model and the step function of the hybrid N4 model are similar to each other. Therefore, the objective functions of both models may be also similar to each other. The average values derived using the different ten random seeds are shown to reduce sampling errors.

4.2.1. Objective functions and CVaR ratios
The left-hand side of Figure 7 shows the objective function values derived by four models. The objective function values of the hybrid N4 model is the largest among three hybrid models because we can control risk and return as the number of decision variables increases. The objective function values of the PwL model is a little bit larger than that of the hybrid N4 model under the condition of the same number of decision variables. However, we need to pay attention to the fact that the differences are slight among these objective function values. It is expected to control risk and return appropriately by employing the piecewise linear investment unit function in terms of wealth as well as the step function in the hybrid N4 model.

In addition, we compare the models using the CVaR ratio in Equation (4.1) expressed

\[\text{CVaR} = \frac{1}{1-\alpha} \int_{\alpha}^{1} \min\{0, \sum_{i=1}^{n} w_i r_i \} \, \text{d}F(x)\]

where $w_i$ is the weight of the $i$-th asset, $r_i$ is the return of the $i$-th asset, and $\alpha$ is the confidence level.

\[\text{CVaR}_{\alpha} = \frac{1}{1-\alpha} \int_{\alpha}^{1} \min\{0, \sum_{i=1}^{n} w_i r_i \} \, \text{d}F(x)\]

The interest rates are path-dependent in the formulation, however we assume the interest rates are deterministic in the numerical analysis for simplicity.
by both risk and return measures to evaluate the efficiency of risk-return trade-off.

\[
\text{CVaR ratio} = \frac{E[W_T] - W_0}{\text{CVaR}} \tag{4.1}
\]

The CVaR ratio measures the excess terminal wealth per unit of CVaR. The higher the investment efficiency is, the larger the ratio is. The right-hand side of Figure 7 shows the CVaR ratios. It is possible to invest in assets efficiently by using the PwL model as well as the hybrid N4 model.

![Figure 7: Objective function value and CVaR ratio](image)

### 4.2.2. Asset allocation

We show the optimal solutions of the hybrid N4 model the PwL model in Table 5. The optimal solutions of the hybrid N4 model are investment weights for each node, while these of the PwL model are the intercept and slope parameters of the piecewise linear function below and above VaR. The cash ratio in the PwL model is derived as one minus the sum of the risky asset ratios.

<table>
<thead>
<tr>
<th>Table 5: Optimal Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hybrid N4 model</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( t = 0 )</th>
<th>( t = 1 )</th>
<th>( t = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Node 1</td>
<td>Node 2</td>
<td>Node 3</td>
</tr>
<tr>
<td></td>
<td>( w_{j0} )</td>
<td>( z^1_{j1} )</td>
<td>( z^2_{j1} )</td>
</tr>
<tr>
<td>DS</td>
<td>14.0%</td>
<td>0.0%</td>
<td>6.9%</td>
</tr>
<tr>
<td>FS</td>
<td>21.0%</td>
<td>0.0%</td>
<td>0.9%</td>
</tr>
<tr>
<td>DB</td>
<td>15.2%</td>
<td>12.1%</td>
<td>70.6%</td>
</tr>
<tr>
<td>FB</td>
<td>49.7%</td>
<td>77.9%</td>
<td>12.5%</td>
</tr>
<tr>
<td>Cash</td>
<td>0.0%</td>
<td>10.0%</td>
<td>10.0%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( t = 0 )</th>
<th>( t = 1 )</th>
<th>( t = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Below VaR</td>
<td>Above VaR</td>
<td>Below VaR</td>
</tr>
<tr>
<td></td>
<td>( a^1_{j1} )</td>
<td>( b^1_{j1} )</td>
<td>( a^2_{j2} )</td>
</tr>
<tr>
<td>DS</td>
<td>14.8%</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>FS</td>
<td>23.1%</td>
<td>-0.04</td>
<td>0.00</td>
</tr>
<tr>
<td>DB</td>
<td>8.9%</td>
<td>-65E+6</td>
<td>65E+4</td>
</tr>
<tr>
<td>FB</td>
<td>53.2%</td>
<td>65E+6</td>
<td>-65E+4</td>
</tr>
</tbody>
</table>

\( w_{j0} = \rho_{j0} z_{j0}/W_0 \), Unit: \( a^a_{j1}(%), b^b_{j1}(%/\text{million yen}) \)

Figure 8 shows the relationship between the amount of wealth and the asset allocation (investment weights) for three models; the hybrid N1 model on the left, the hybrid N4 model...
model in the middle and the PwL model on the right. The order of assets described in
the graphs is the DS, FS, DB and FB from underneath. The weights at time 1 and time
2 depend on the amount of wealth. The weights at time 0 in the three graphs on the top
of Figure 8 are different from the graphs of time 1 and time 2. The PwL functions of the
weights are divided by $W_0 - \text{VaR}$ shown by the black lines on the graphs of time 1 and time
2. The value of $W_0 - \text{VaR}$ is called 'VaR level' or 'VaR point' hereafter. The black lines
also show the VaR levels ($W_0 - \text{VaR}$) on the graphs of the hybrid N4 model. We discuss the
characteristics about the asset weights in what follows.

![Figure 8: Asset allocation](image)

(1) **Time 0**

The DB which is the lowest risky asset is more invested than the DS and FS which are
higher risky assets in the hybrid N1 model. This reason is in what follows. The adequate
investment decision for all paths on average is made after time 1 because the state-dependent
decision in terms of wealth cannot be made in the hybrid N1 model. As a result, the risky
investment cannot be made at time 0, and the most conservative investment strategy is
implemented. This is different from other models. On the other hand, the asset mix of
the PwL model is similar to that of the hybrid N4 model. Less investment in the DB and
more investments in the DS, FS and FB in the both models are made than the investment
in the hybrid N1 model. The reason is that the state-dependent decisions can be made
after time 1 for both models even though their investment unit functions are different each
other: step function for the hybrid model and piecewise linear function for the PwL model.
If investors have a larger amount of wealth, investors adopt the strategy of taking risk to gain higher return at time 1 because it is possible to make the optimal state-dependent investment decision. If investors have a smaller amount of wealth at time 1, investors adopt the strategy of reducing risk and aiming for a recovery of wealth little by little. Consequently, riskier asset mix policy is implemented than that of the hybrid N1 model.

(2) Time 1

In the hybrid N1 model, the adequate investment decision is made on average for all paths, and therefore the most conservative investment strategy is implemented. The DB is mainly invested and the weights of stocks are about 10% because stocks are risky assets, and thus asset mix at time 1 is almost the same as time 0.

We have the similar asset mix for both the hybrid N4 model and the PwL model. As the amount of wealth becomes large, the weights of stocks (DS and FS) increase and the weight of bonds (DB and FB) decreases. We can afford to invest in the stocks because we have much wealth, and attempt to aim the high return.

When the amount of wealth is small below VaR, the optimal weights of DB and FB jump like step functions at the VaR point. The slope parameter of DB ($b_{131} = 65 \times 10^4$) is positively large, that of of FB ($b_{111} = -65 \times 10^4$) is negatively large. We need to constrain the slope parameters to prevent the optimal weights from jumping at the VaR point.

(3) Time 2

The asset allocations at time 2 are similar to time 1 for all of the models, but the amount of wealth affects the more allocation change at time 2 than time 1. Compared with the results at time 1, stocks are more invested below VaR level, while bonds are more invested above VaR level. The asset mix is determined so that the amount of wealth can attempt to exceed VaR level in the case that the amount of wealth is below the VaR point and it can keep exceeding VaR level above the VaR point.

4.3. Sensitivity analysis

The objective function values and the CVaR ratios of the PwL model are better than but almost the same as the hybrid N4 model in the base case. We conduct the sensitivity analysis for the two kinds of parameters; risk averse coefficient ($\gamma$) and autocorrelation ($c$), as follows.

$\gamma$ : three kinds of risk averse coefficients ($\gamma = 0.6, 0.8, \infty$) \(^{17}\)

$c$ : five kinds of autocorrelations ($c = 0, 0.25, 0.5, 0.75, 1$)

We solve the problem with no autocorrelation in the case of $c = 0$, and estimated autocorrelation in the case of $c = 1$. The average of values calculated using the different ten random seeds are shown to reduce sampling errors as well as the base case.

4.3.1. Objective function

The objective function values of 15 combinations of $\gamma$ and $c$ are shown in Figure 9. The CVaR minimization problem needs to be formulated to solve the problem for $\gamma \rightarrow \infty$. Therefore, the larger values are better for $\gamma = 0.6$ and 0.8, and the smaller value is better for risk minimization problem ($\gamma \rightarrow \infty$) because the objective function is CVaR. We obtain the similar results to the base case.

As the autocorrelation parameter ($c$) becomes large, the objective function value becomes better. This result shows that we can take an appropriate investment strategy for asset mix

\(^{17}\)The risk averse coefficients ($\gamma$) are non-negative. The expected terminal wealth is maximized for $\gamma = 0$, and the CVaR is minimized for $\gamma \rightarrow \infty$. As the risk-averse coefficient becomes large, the risk-averse and conservative strategy is adopted.
to control the risk-return tradeoff according to the existence of autocorrelation.

As the risk-averse coefficient ($\gamma$) becomes large, the differences among the models become small. In the risk minimization problem where $\gamma \to \infty$, the CVaR of the PwL model is the smallest, and the CVaR of the hybrid model decreases inversely with the number of decision nodes regardless of the autocorrelation. The reason is that the conservative strategy tends to be adopted as the coefficient $\gamma$ becomes large. As the results, there are little differences among the models.

4.3.2. Asset allocation
We examine the investment proportion at each time for three kinds of the autocorrelations, and describe the relationship between those and the amount of wealth with $\gamma = 0.6$.

1. Asset allocation at time 0
We show the asset allocation at time 0 in Figure 10.

In the case of $c = 0$, there are little different among four models. But as the autocorrelation parameter $c$ becomes large, the weights of stocks (DS and FS) become large and we can find the explicit difference among these models. As stated above, the autocorrelation parameters affect the asset mix so that the objective function values can be improved. We can determine the optimal weights in accordance with the amount of wealth at time 1 and 2 in the hybrid N4 model and the PwL model.

2. Asset allocation at time 1 and time 2
We show asset mix at time 1 in Figure 11 and time 2 in Figure 12. The black bold lines show the VaR level in the graphs. The graphs on the top are shown for the hybrid
N4 model and the graphs on the bottom are shown for the PwL model. The graphs on the left-hand side are shown for $c = 0$ and the graphs on the right-hand side are shown for $c = 0.5$. Asset mixes vary with the different autocorrelation, and we examine the difference of the investment decisions.

When the amount of wealth is above VaR point, the weights of the PwL model are also similar to the weights of the hybrid N4 model as well as the results of the base case. We have typical weight functions for the PwL model. The weights of stocks (DS and FS) become large, and the weights of bonds become small as the parameter $c$ is large. When the parameter $c$ is large, we can invest in the stocks, or riskier assets because we take risk by employing autocorrelations among assets. In addition, the weights of stocks at time 2 are smaller than those weights at time 1. We do not need to take risk at time 2 when the amount of wealth is above VaR.

When the amount of wealth is below VaR point, the optimal weights of DB is more than 80% at time 1 and 70% at time 2 in the hybrid N4 model. The decisions are conservative because four nodes in the hybrid N4 model are divided at 25%, 50%, 75% points, and the simulated paths below 80% VaR belong to the same node. On the other hand, the optimal weights of FB, DS, and FS in the PwL model become larger than those weights in the hybrid N4 model. We can control investment decisions in the PwL model.

Figure 11: Asset allocation at time 1
5. Concluding remarks

In this paper, we propose the piecewise linear model for state-dependent asset allocation with CVaR to solve the multi-period portfolio optimization problem with the Monte Carlo simulation. We conduct the sensitivity analysis for the several risk averse coefficients and autocorrelations, and we examine the characteristics of the model.

In the future research, we need to study the following problems.

1. The piecewise linear function proposed in this paper is kinked only at the VaR. We may need to increase the number of kinked points to express the state-dependent function precisely.

2. The piecewise linear function is dependent on only the amount of wealth. We may need to introduce the other state variables to express the state-dependent function.

3. We need to evaluate the performance by using the practical or out-of-sample data.

4. We attempt to examine the results in the various cases of the complicated stochastic process, the relationship between the number of paths and sampling error, and so on.

5. We need to clarify the forms of the state-dependent functions for the other risk measures by using the hybrid model with a lot of decision nodes.

References


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