A Hybrid Simulation/Tree Stochastic Optimization Model for Dynamic Asset Allocation *

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Abstract

Asset allocation decisions are critical for investors with diversified portfolios. Institutional investors must manage their strategic asset mix over time to achieve favorable returns subject to various uncertainties, policy and legal constraints, and other requirements. In order to determine the asset mix explicitly, we can use a multi-period portfolio optimization model.

The notion of scenarios is typically employed for modeling random parameters in the multi-period stochastic programming(MSP) model, and the path of uncertainties is revealed as a scenario tree. On the other hand, a simulated path model is proposed and formulated as a linear programming model by Hibiki[4, 5]. The path of uncertainties is revealed as simulated paths associated with any stochastic process, and we can get a better accuracy of uncertainties by using simulated paths rather than a scenario tree. Two kinds of stochastic optimization models involve trading off the appropriate decisions and the accuracy of uncertainties.

In this paper, we propose a hybrid optimization model by using simulated paths and decision tree in order to do both describing the accuracy of uncertainties and making the conditional decisions. We have sample paths associated with asset returns using Monte Carlo simulation as used in the simulated path model, while we allow the model to expand the decision space and to make the conditional decisions as used in the scenario tree model. We test some cases using numerical examples We find that we can control the risk and return of the terminal wealth in some degree directly by using this model.

The hybrid model can be formulated as its compact representation form to reduce the program size. We find that we can decrease its computation time, according to some numerical tests.

*The hybrid optimization model is originally proposed by Hibiki[6, 7](written in Japanese). This paper is the revised version in English, and includes a modification of Hibiki[7] and an additional idea discussed in Section 5.

1 Introduction

Investors need to maximize the expected utility of returns from their investment portfolios, or to minimize the risk exposure of returns subject to the required expected returns. They must decide optimal portfolios in securities in order to meet their satisfaction. This paper discusses optimal dynamic investment policies for investors, who make the investment decisions in each of asset categories over time. This problem is called dynamic asset allocation.

Asset allocation decisions are critical for investors with diversified portfolios. Institutional investors must manage their strategic asset mix over time to achieve favorable returns subject to various uncertainties, policy and legal constraints, and other requirements. In order to determine the asset mix explicitly, we can use a multi-period portfolio optimization model.

Critical issues for stochastic modeling involve the handling of uncertainties and investment decisions. The decisions have to be made independent upon knowing actual paths that will occur. Thus we must define decision variables and a set of constraints to prevent the optimization model from anticipating the future. In addition, we need sufficient paths of uncertainties to get a better accuracy with respect to the future possible events. We describe the characteristics of two models in order.

The notion of scenarios is typically employed for modeling random parameters in the multi-period stochastic programming(MSP) model, and the path of uncertainties is revealed as a scenario tree. This model is based on the expansion of the decision space, taking into account the conditional nature of the scenario tree. Conditional decisions are made at each node. To assume that a representative set of scenarios is constructed which covers the set of possibilities to a sufficient degree, the numbers of decision variables and constraints appearing in the scenario tree may grow exponentially. (See Mulvey and Ziemba [9, 10] in detail.)

On the other hand, dynamic stochastic control offers an alternative to stochastic programming for setting dynamic investment strategies. The basic framework of this model is originally proposed by Merton[8] and Samuelson[11]. In general, stochastic control forms a mesh over the state space, instead of discretizing the scenarios. However, we can also assume the discretizations, or discrete time and discrete scenarios, and then the path of uncertainties is revealed as simulated paths associated with any stochastic process. We take advantage of simulated paths instead of a scenario tree to describe a better accuracy of uncertainties in the multiple assets problem. In this framework, we couple that with a specific form of investment decisions at each period. The fixed-mix strategy fits this framework, and investment decisions at each period could be limited to a fixed-proportion rule, for example, and hence, this method reduces the decision space to a set of policies. This approach can be formulated as a stochastic programming model. But this formulation includes non-convex (non-linear) constraints, and therefore it is difficult to solve the problem and to find the global optimal solutions.

An alternative model using simulated paths is proposed and formulated as a linear programming model by Hibiki[4, 5]. We name this model “simulated path model” in this paper. The linear programming model is formulated by using a fixed-unit rule instead of a fixed-proportion rule. This formulation can be simply implemented and solved very fast by using sophisticated mathematical programming softwares.

Two stochastic programming models such as the scenario tree model and the simulated path model can be used to solve the multi-period optimization problems in practice.

Table 1: Comparison between two models

	scenario tree model	simulated path model
accuracy of uncertainties	scenario tree	simulated paths
investment decisions	every nodes of all periods	all periods

Two kinds of stochastic optimization models can be used to solve large-scale problems. However, they involve trading off the appropriate decisions and the accuracy of uncertainties. We can use the scenario tree model in the case that we wish to have an appropriate (conditional) decision, while we can use the simulated path model in the case that we wish to have the accuracy of uncertainties. The model selection depends on what we regard as most important.

In this paper, we propose a hybrid optimization model by using simulated paths and the decision tree in order to do both describing the accuracy of uncertainties and making the conditional decisions. We have sample paths associated with asset returns using Monte Carlo simulation as used in the simulated path model, while we allow the model to expand the decision space and to make the conditional decisions as used in the scenario tree model.

The paper is organized as follows. The hybrid model can be formulated as the modification from the simulated path model, and then Section 2 introduces the simulated path model. Section 3 presents the concept and formulation of the hybrid model. Section 4 shows numerical examples. Section 5 translates the formulation of the proposed model into its compact representation forms. Section 6 provides some concluding remarks and our future research.

2 Simulated path model

2.1 Preparation

An asset return is a random parameter, and its process is expressed by a stochastic differential equation, or a time series model. We can sample simulated paths of each asset return on each simulation trial. An example of simulated paths is shown as in Figure 1.

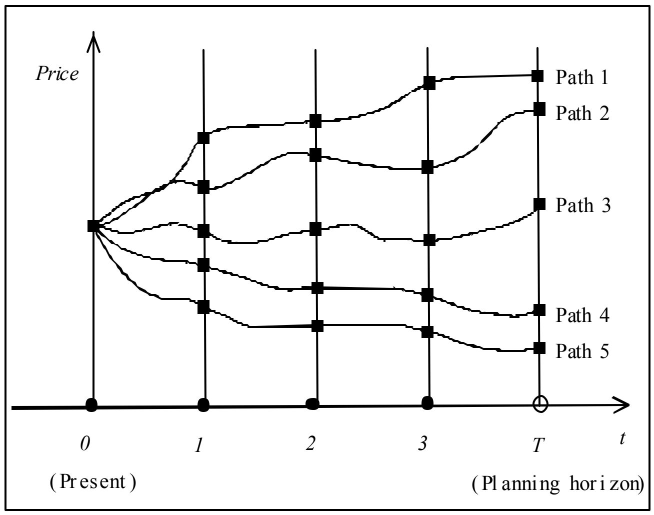

Figure 1: Simulated path

Hibiki[4, 5] develop the simulated path model by using Monte Carlo simulation to describe the accuracy of uncertainties in a multi-period optimization framework, where discrete values of asset return are generated by Monte Carlo simulation.

Investment units are used for expressing the MSP model. The solution to the asset allocation decision of this model provides the recommended units, however they are defined as the linear function

$$h^{(i)}(z_{jt}) = a^{(i)}_{jt} z_{jt} + b^{(i)}_{jt}, \tag{1}$$

where z_{jt} is the base decision (unit) in the fixed-unit rule [1] , and it is used as the decision variable in the model. The base investment units of risky assets must be fixed on all paths at each time to keep the non-anticipativity condition [2] , however they can be various over time.

$a^{(i)}_{jt}$ and $b^{(i)}_{jt}$ are parameters associated with investment units, and are set according to the corresponding investment strategy [3] . If we want to increase investment units when price goes up, and to decrease when price goes down, we may set this parameter, such as

$$a^{(i)}_{jt} = 1 + \mu^{(i)}_{jt}, \text{ and } b^{(i)}_{jt} = 0$$

where $\mu^{(i)}_{jt}$ is the rate of return of asset j of path i in period t. On the other hand, if we want to

[1] The term "fixed-" does not mean "buy and hold strategy", and "constant rebalance strategy".

[2] Cash does not need to be fixed in each path. Even if cash is different in each path, we can keep the non-anticipativity condition because cash return is risk-free in the beginning of the period.

[3] Investment units are directly used as decision variables in the original simulated path model[4, 5]; that is, $h(z_{jt}) = z_{jt}$, where z_{jt} is the investment unit of asset j at time t.

decrease investment units when price goes up, and to increase when price goes down, we may set this parameter, such as

$$a_{jt}^{(i)} = \frac{1}{1+\mu_{jt}^{(i)}}, \text{ and } b_{jt}^{(i)} = 0.$$

2.2 Model formulation

We invest n risky assets and cash. The investment is made at time 0(present), and time T is the planning horizon.

We determine the asset mix by using two kinds of measure; the expected terminal wealth and the first-order lower partial moment(LPM_1) of terminal wealth[1, 3]. The former corresponds to the return measure, and the latter corresponds to the risk measure. The lower partial moment is one of the downside-risk measures, and involve the tail of the relevant distribution of wealth below target wealth. Only the left-hand tail of the distribution of wealth is used in the calculation.

Computationally, the LPM for an empirical(discrete) distribution of terminal wealth, $W_T^{(i)}$, with a target wealth, W_G, is described by: [4]

$$LPM_k \equiv \frac{1}{I}\sum_{i=1}^{I}\left|W_T^{(i)} - W_G\right|_-^k \tag{2}$$

where I is the number of samples, and $|a|_- = \max(-a, 0)$. The risk measure becomes the first-order LPM for $k = 1$ [5] .

We can formulate as a linear programming problem by using the first-order LPM, and we can easily a large-scale problem in practical use. We list notations in this model.

(1) Parameters

- I : number of simulated paths
- ρ_{j0} : price of risky asset j at time 0, $(j = 1, \ldots, n)$
- $\rho_{jt}^{(i)}$: price of risky asset j of path i at time t,$(j = 1, \ldots, n;\ t = 1, \ldots, T;\ i = 1, \ldots, I)$
- r_0 : interest rate in period 1 (the rate at time 0 is used).
- $r_{t-1}^{(i)}$: interest rate of path i in period t (the rate at time $t-1$ is used), $(t = 1, \ldots, T;\ i = 1, \ldots, I)$
- W_0 : initial wealth
- W_G : target wealth at the planning horizon
- W_E : required expected wealth at the planning horizon

(2) Decision variables

[4]The LPM for a continuous distribution of terminal wealth $\tilde{W}_T$ is described as follows:

$$LPM_k \equiv \int_{-\infty}^{W_G} (W_G - \tilde{W}_T)^k f(\tilde{W}_T) d\tilde{W}_T$$

[5]The global optimal solution can be guaranteed because of the convex programming problem even if the k-th order moments(LPM_k) are used.

z_{jt} : base investment units of risky asset j at time t, $(j = 1, \dots, n;\ t = 0, \dots, T-1)$

v_0 : cash at time 0

$v_t^{(i)}$: cash of path i at time t,$(t = 0, \dots, T-1;\ i = 1, \dots, I)$

$q^{(i)}$: shortfall below the target wealth of path i at the planning horizon, $(i = 1, \dots, I)$

We denote investment units by

$$h^{(i)}(z_{jt}) = a_{jt}^{(i)} z_{jt} + b_{jt}^{(i)},\ (j = 1, \dots, n;\ t = 1, \dots, T-1;\ i = 1, \dots, I) \tag{3}$$

as described early.

We formulate the MSP model as follows:

$$\textbf{Minimize} \qquad \frac{1}{I} \sum_{i=1}^{I} q^{(i)} \tag{4}$$

subject to

$$\sum_{j=1}^{n} \rho_{j0} z_{j0} + v_0 = W_0 \tag{5}$$

$$\sum_{j=1}^{n} \rho_{j1}^{(i)} z_{j0} + (1 + r_0) v_0 = \sum_{j=1}^{n} \rho_{j1}^{(i)} h^{(i)}(z_{j1}) + v_1^{(i)}, \quad (i = 1, \dots, I) \tag{6}$$

$$\sum_{j=1}^{n} \rho_{jt}^{(i)} h^{(i)}(z_{j,t-1}) + \left(1 + r_{t-1}^{(i)}\right) v_{t-1}^{(i)} = \sum_{j=1}^{n} \rho_{jt}^{(i)} h^{(i)}(z_{jt}) + v_t^{(i)},$$

$$(t = 2, \dots, T-1;\ i = 1, \dots, I) \tag{7}$$

$$\frac{1}{I} \sum_{i=1}^{I} \left\{ \sum_{j=1}^{n} \rho_{jT}^{(i)} h^{(i)}(z_{j,T-1}) + \left(1 + r_{T-1}^{(i)}\right) v_{T-1}^{(i)} \right\} \geq W_E \tag{8}$$

$$\left\{ \sum_{j=1}^{n} \rho_{jT}^{(i)} h^{(i)}(z_{j,T-1}) + \left(1 + r_{T-1}^{(i)}\right) v_{T-1}^{(i)} \right\} + q^{(i)} \geq W_G,\ (i = 1, \dots, I) \tag{9}$$

$$z_{jt} \geq 0, \quad (j = 1, \dots, n;\ t = 0, \dots, T-1)$$

$$v_0 \geq 0$$

$$v_t^{(i)} \geq 0, \quad (t = 1, \dots, T-1;\ i = 1, \dots, I)$$

$$q^{(i)} \geq 0, \quad (i = 1, \dots, I)$$

3 The hybrid model

3.1 Modelling for conditional decisions

We propose the model, which makes the conditional decisions under the framework using simulated paths. We make up several bundles of simulated paths at each time, and we have the fixed-unit strategies for risky assets at each bundles. We call the bundles "fixed-decision nodes", and we generate the decision tree where conditional decisions are made at each nodes. we call this tree the "extended decision tree" to distinguish from the scenario tree.

Suppose 20 simulated paths over three periods are represented as in Figure 2. For simplicity, we assume to make up three bundles of 20 paths at time 1, and more two bundles of paths within each bundles at time 2. In total, we have six bundles at time 2.

The bundling procedure is illustrated schematically in Figure 2. First, we generate 20 paths associated with asset returns over the planning period. Next, 20 paths are classified into three clusters by asset returns in period 1 under the appropriate criterion [6] . We have nine, six, and five paths in each cluster, named node "A", "B", and "C". The conditional fixed-decisions are made at each node, respectively.

Finally, the same procedure is carried out in every nodes at time 1 in period 2. Specifically, nine paths through node "A" are classified into two clusters by asset returns in period 2. We have five and four paths in each cluster. Similarly, six paths through node "B", and five paths through node "C" are classified into two clusters, respectively. We have two and four paths in each cluster from node "B", and have three and two paths in each cluster from node "C". We have six kinds of conditional decisions at time 2.

Three black points at time 1, and six points at time 2 in Figure 2(3) represent the nodes associated with the conditional decisions in the extended decision tree.

If paths are not classified into two or more clusters, the hybrid model reduces itself to the simulated path model. If each asset has the same return on the paths through any node, the hybrid model reduces itself to the scenario tree model. The hybrid model is therefore the comprehensive model including the simulated path model and the scenario tree model.

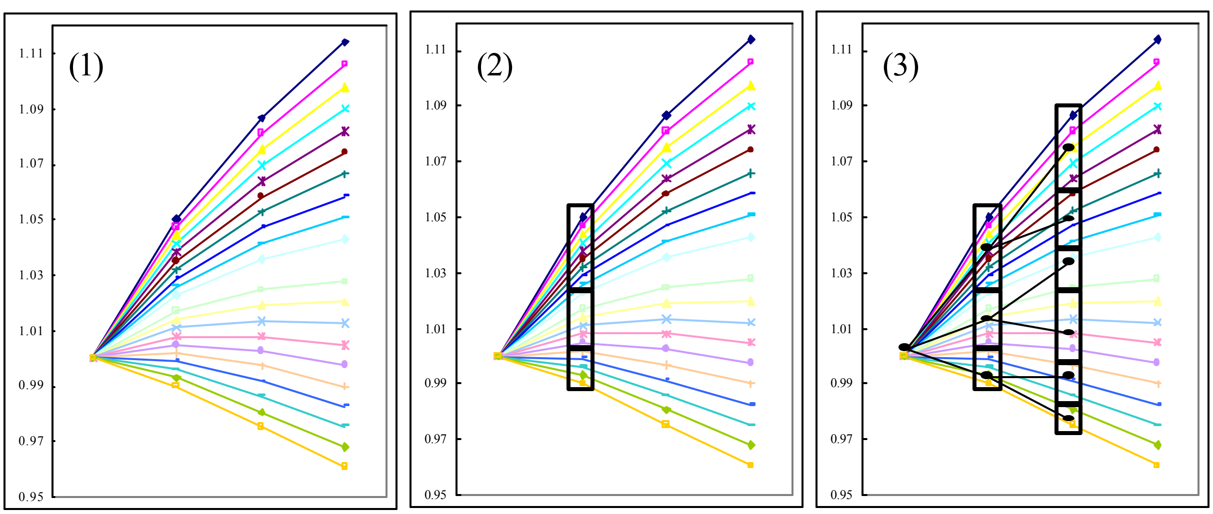

Figure 2: Extended decision tree

3.2 Sequential clustering method

We propose the technique of generating the extended decision tree, which we call the sequential clustering method. This method is applied to the data set of simulated paths over the planning

[6] We may use any classifying method. In this study, we use sequential clustering method, which is also proposed in this paper. Refer to next subsection.

period by using the well-known hierarchical clustering method [7] in each period sequentially. Generated clusters represent the fixed-decision nodes. It can be implemented based on similarities calculated by distances between sampled return vectors. Specifically, we show the procedure as follows;

Step 1 Clustering method is implemented by the distances between sampled return vectors in period 1. We have several fixed-decision nodes whose number is predetermined.

Step 2 Likewise, clustering method is implemented in each node in period 1, by the distances between sampled return vectors in period 2 on the path through each node in period 1. We have several nodes in period 2.

Step 3 This procedure is continued, working forward until the planning time horizon T is reached and hence sampled returns are not required to be clustered.

3.3 Model formulation

(1) Set

S_t : set of fixed-decision nodes at time t $(s \in S_t)$

V_t^s : set of paths including any fixed-decision node s at time t $(i \in V_t^s)$

(2) Parameter

We have the same parameters as in the simulated path model. Refer to the parameters in Subsection 2.2(1).

(3) Decision variables

z_{j0} : investment units for asset j and time 0 $(j = 1, \ldots, n)$

z_{jt}^s : base investment units for asset j, time t, and node s $(j = 1, \ldots, n;\ t = 1, \ldots, T-1;\ s \in S_t)$. We denote investment units by $h^{(i)}(z_{jt}^s) = a_{jt}^{(i)} z_{jt}^s + b_{jt}^{(i)}$ as in the simulated path model.

v_0, $v_t^{(i)}$, and $q^{(i)}$ are the same variables as in the simulated path model. (Refer to Subsection 2.2(2)).

[7] We use Ward method, which is one of the hierarchical clustering methods. Other method are nearest neighbor method, furthest neighbor method, group average method, and so on. It is said that Ward method is superior to other methods in practical use.

(4) Formulation

We formulate the hybrid model as follows:

$$\textbf{Minimize} \quad \frac{1}{I}\sum_{i=1}^{I} q^{(i)} \tag{10}$$

subject to

$$\sum_{j=1}^{n} \rho_{j0} z_{j0} + v_0 = W_0 \tag{11}$$

$$\sum_{j=1}^{n} \rho_{j1}^{(i)} z_{j0} + (1+r_0)v_0 = \sum_{j=1}^{n} \rho_{j1}^{(i)} h^{(i)}(z_{j1}^{s}) + v_1^{(i)}, \quad (s \in S_1;\ i \in V_1^s) \tag{12}$$

$$\sum_{j=1}^{n} \rho_{jt}^{(i)} h^{(i)}(z_{j,t-1}^{s'}) + \left(1+r_{t-1}^{(i)}\right) v_{t-1}^{(i)} = \sum_{j=1}^{n} \rho_{jt}^{(i)} h^{(i)}(z_{jt}^{s}) + v_t^{(i)}, \qquad (t=2,\ldots,T-1;\ s \in S_t;\ i \in V_t^s) \tag{13}$$

$$W_T^{(i)} = \sum_{j=1}^{n} \rho_{jT}^{(i)} h^{(i)}(z_{j,T-1}^{s'}) + \left(1+r_{T-1}^{(i)}\right) v_{T-1}^{(i)}, \quad (s' \in S_{T-1};\ i \in V_{T-1}^{s'}) \tag{14}$$

$$\frac{1}{I}\sum_{i=1}^{I} W_T^{(i)} \geq W_E \tag{15}$$

$$W_T^{(i)} + q^{(i)} \geq W_G, \quad (i=1,\ldots,I) \tag{16}$$

$$z_{j0} \geq 0, \quad (j=1,\ldots,n) \tag{17}$$

$$z_{jt}^{s} \geq 0, \quad (j=1,\ldots,n;\ t=1,\ldots,T-1;\ s \in S_t) \tag{18}$$

$$v_0 \geq 0 \tag{19}$$

$$v_t^{(i)} \geq 0, \quad (t=1,\ldots,T-1;\ i=1,\ldots,I) \tag{20}$$

$$q^{(i)} \geq 0, \quad (i=1,\ldots,I)$$

$W_T^{(i)}$ is the terminal wealth for time t and path i $(i=1,\ldots,I)$. We denote s' in Equation (13) the decision node at time $t-1$ connected with the node at time t. Both sides of Equations (12), (13) show the wealth of path i at time t . The problem size is very large, but it has very sparse matrix. The relationship between the number of path and non-zero elements in the constraints to the hybrid model is as in Figure 3.

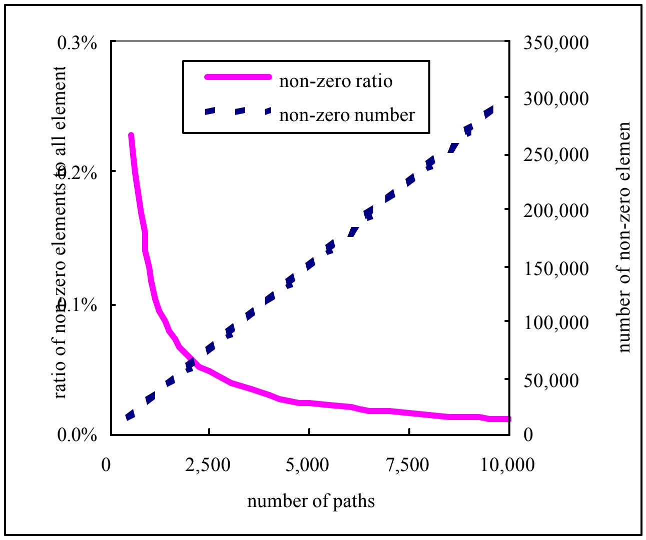

Figure 3: Relationship between the number of paths and non-zero elements($n = 3$, $T = 4$)

4 Numerical Examples

We test the hybrid model using numerical examples. Three periods model is solved. The number of simulated paths is 1,000. The number of constraints except non-negative constraints, and the number of decision variables are about four thousands, respectively. All of the problems are solved using NUOPT(Ver. 4) [8] . It takes about 3.24 seconds to solve three branching problem on Windows 98 personal computer which has 1500 MHz CPU and 512MB memory.

The number of assets is four; stock, bond, CB(convertible bond), and cash.

Table 2 shows the summary statistics(expected rate of return, standard deviation and correlation matrix of rate of return) calculated by the available market data; Nikko stock performance index (TSE 1), Nikko bond performance index, Nikko CB performance index, and call rate.

The rate of return $\mu_{jt}^{(i)}$ is generated by these statistics as follows.

① The rate of return of asset j in period t is normally distributed with mean $\overline{\mu}_{jt}$ and standard deviation σ_{jt}, and it is generated by:

$$\mu_{jt}^{(i)} = \overline{\mu}_{jt} + \sigma_{jt}\varepsilon_{jt}^{(i)},$$

where $\varepsilon_{jt}^{(i)}$ is a random sample from a multi-variate standardized normal distribution.

② The random variable ε_{jt} $(j = 0, \ldots, n;\ t = 1, \ldots, T)$ follows that

$$\varepsilon_{jt} \sim N(\mathbf{o}, \Sigma),$$

where Σ is $(n+1)T \times (n+1)T$ correlation matrix.

Asset 0 is the interest rate, and $\mu_{0t}^{(i)}$ is the change rate of interest rate. The call rate $r_t^{(i)}$ is calculated by:

$$r_1^{(i)} = r_0 \times \left(1 + \mu_{01}^{(i)}\right),$$

$$r_t^{(i)} = r_{t-1}^{(i)} \times \left(1 + \mu_{0t}^{(i)}\right),\ (t = 2, \ldots, T-1).$$

[8] NUOPT is the mathematical programming software, and it is developed by Mathematical System, Inc.

Table 2: Summary statistics

		cash			stock			bond			CB		
		1	2	3	1	2	3	1	2	3	1	2	3
expected return		−0.087	−0.081	−0.089	0.848	0.867	0.843	0.625	0.623	0.645	0.786	0.780	0.786
standard deviation		0.780	0.784	0.778	5.571	5.582	5.595	1.372	1.372	1.353	3.543	3.541	3.538
correlation		1	2	3	1	2	3	1	2	3	1	2	3
cash	1	1.000	−0.091	0.073	−0.101	0.000	−0.032	−0.238	0.008	0.090	−0.146	−0.044	−0.052
	2	−0.091	1.000	−0.092	0.045	−0.094	−0.007	−0.183	−0.237	0.011	−0.012	−0.144	−0.047
	3	0.073	−0.092	1.000	0.016	0.042	−0.091	−0.166	−0.188	−0.221	−0.062	−0.017	−0.138
stock	1	−0.101	0.045	0.016	1.000	0.022	−0.031	0.145	−0.173	−0.096	0.761	0.042	−0.045
	2	0.000	−0.094	0.042	0.022	1.000	0.018	0.085	0.144	−0.170	0.019	0.760	0.0414
	3	−0.032	−0.007	−0.091	−0.031	0.018	1.000	0.077	0.085	0.141	0.011	0.019	0.760
bond	1	−0.238	−0.183	−0.166	0.145	0.085	0.077	1.000	0.130	−0.108	0.327	0.202	0.065
	2	0.008	−0.237	−0.188	−0.173	0.144	0.085	0.130	1.000	0.137	−0.114	0.327	0.204
	3	0.090	0.011	−0.221	−0.096	−0.170	0.141	−0.108	0.137	1.000	−0.180	−0.109	0.321
CB	1	−0.146	−0.012	−0.062	0.761	0.019	0.011	0.327	−0.114	−0.180	1.000	0.092	−0.068
	2	−0.044	−0.144	−0.017	0.042	0.760	0.019	0.202	0.327	−0.109	0.092	1.000	0.093
	3	−0.052	−0.047	−0.138	−0.045	0.041	0.760	0.065	0.204	0.321	−0.068	0.093	1.000

Initial prices of stock, bond, and CB can be assumed to be 1 without loss of generality. The initial call rate is 0.44%. The initial wealth and the target wealth are 100 million yen. For simplicity, we set the parameters associated with investment unit as $a_{jt}^{(i)} = 1$, and $b_{jt}^{(i)} = 0$; that is, $h(z_{jt}^s) = z_{jt}^s$. Any fixed-decision node is assumed to branch into two, three, four, or five nodes, and the number of fixed-decision nodes are as follows:

branch	time 1	time 2
2	2	4
3	3	9
4	4	16
5	5	25

We test eight cases where the objective functions and their related constraints to the required expected terminal wealth are various as in Table 3.

Table 3: Test case

unit: 10 thousand yen

case 1	Minimization of the risk
case 2	Minimization of the risk subject to $W_E = 10,180$
case 3	Minimization of the risk subject to $W_E = 10,195$
case 4	Minimization of the risk subject to $W_E = 10,210$
case 5	Minimization of the risk subject to $W_E = 10,225$
case 6	Minimization of the risk subject to $W_E = 10,240$
case 7	Minimization of the risk subject to $W_E = 10,255$
case 8	Maximization of the expected terminal wealth

While we use the formulation shown in Subsection 3.3 for case 2 to 7, we should use the following formulations for case 1 and case 8, respectively [9] .

(case 1)

$$\begin{array}{ll} \textbf{Minimize} & \frac{1}{I}\sum_{i=1}^{I} q^{(i)} \\ \textbf{subject to} & \text{Equations } (11)-(14),(16)-(21) \end{array}$$

(case 8)

$$\begin{array}{ll} \textbf{Maximize} & \frac{1}{I}\sum_{i=1}^{I} W_T^{(i)} \\ \textbf{subject to} & \text{Equations } (11)-(14),(16)-(21) \end{array}$$

[9] The values of expected wealth for case 1 and case 8 in each branching number are as follows:

unit: 10 thousand yen

branch	case 1	case 8
2	10,154.1	10,274.7
3	10,161.1	10,280.4
4	10,166.1	10,293.6
5	10,174.3	10,301.5

4.1 Example 1 : Different requirements of terminal wealth

(1) Optimal solutions

We show the results for the cases that branching is three. Optimal investment units, v_0^*, $v_t^{(i)*}$ for cash, z_{jt}^s for risky assets, can be obtained as in Table 4. Due to large number of intermediate cash decisions($v_t^{(i)*}$), we show only the average cash value for each node, $\overline{v}_t^{s*}$.

Table 4: Optimal investment units

case 1

cash			stock			bond			CB		
0	1	2	0	1	2	0	1	2	0	1	2
8075.9	7762.1	6542.5	117.1	0.0	0.0	1109.9	1415.1	3496.0	697.1	856.4	0.2
		6340.8			75.4			3266.7			375.3
		8544.8			0.0			0.0			1516.5
	7995.0	6759.4		0.0	0.2		0.0	0.1		1993.0	3148.0
		9032.0			1033.3			0.2			0.3
		9507.3			555.5			0.0			0.0
	7882.8	9331.7		461.5	67.6		1689.8	633.3		0.0	0.0
		6527.0			99.3			1.1			3406.5
		7818.3			0.0			1892.1			341.6

case 3

cash			stock			bond			CB		
0	1	2	0	1	2	0	1	2	0	1	2
5656.9	9188.5	461.8	6.2	0.7	198.1	3167.7	211.0	8915.2	1169.3	648.8	431.2
		5671.7			37.7			4139.5			212.1
		7574.0			0.2			1.0			2430.7
	6277.0	6217.1		35.5	20.1		0.3	1.8		3614.0	3718.9
		8663.3			1406.7			6.5			5.2
		9871.6			185.3			1.4			0.3
	3941.1	4685.6		638.0	0.1		5442.9	5291.1		0.1	0.1
		718.8			2987.8			0.4			6267.0
		9627.7			2.0			439.7			5.7

case 5

cash			stock			bond			CB		
0	1	2	0	1	2	0	1	2	0	1	2
2261.0	8579.1	302.6	312.4	0.0	0.2	5463.7	14.6	9105.6	1962.9	1460.5	581.5
		3702.6			349.2			6007.5			0.8
		6510.9			0.0			0.1			3484.5
	1916.5	4067.3		0.5	0.7		0.0	0.2		7863.5	5946.9
		7996.3			2149.0			0.7			0.5
		8715.8			1263.9			0.1			0.0
	174.2	204.6		338.2	0.0		9430.5	9707.5		0.0	0.0
		759.8			2878.7			0.0			6319.4
		7838.0			0.1			2182.0			0.5

case 7

cash			stock			bond			CB		
0	1	2	0	1	2	0	1	2	0	1	2
0.0	4461.2	525.2	473.3	0.0	0.1	5734.9	1692.7	9193.1	3791.7	3878.0	217.4
		223.9			418.0			7420.5			1968.1
		461.4			0.0			0.0			9354.0
	324.2	334.4		1010.2	20.3		0.0	0.2		8437.9	9489.4
		4271.4			5660.3			0.4			0.2
		6849.3			3097.9			0.2			0.1
	284.3	297.2		116.8	0.0		9456.0	9547.6		0.0	0.0
		899.4			9031.3			0.0			0.3
		215.6			0.1			9623.6			6.2

Average investment proportions can be calculated by the optimal(recommended) investment units as follows:

$$\text{cash} \ : \ \frac{v_0^*}{W_0}, \ \frac{\overline{v}_t^{s*}}{\overline{W}_t^{s*}}, (t=1,2)$$

$$\text{risky asset} \ : \ \frac{\rho_{j0} z_{j0}^*}{W_0}, (j=1,2,3), \ \frac{\overline{\rho}_{jt}^s z_{jt}^{s*}}{\overline{W}_t^{s*}}, (j=1,2,3; \ t=1,2; \ s \in S_t)$$

The average values of asset price, cash, and wealth are described by:

$$\text{Average asset price in each node} : \overline{\rho}_{jt}^s \ = \ \frac{1}{\overline{V}_t^s} \sum_{i \in V_t^s} \rho_{jt}^{(i)}$$

$$\text{Average cash in each node}: \overline{v}_t^s \ = \ \frac{1}{\overline{V}_t^s} \sum_{i \in V_t^s} v_t^{(i)}$$

$$\text{Expected wealth in each node}: \overline{W}_t^s \ = \ \frac{1}{\overline{V}_t^s} \sum_{i \in V_t^s} W_t^{(i)}$$

where V_t^s denotes the number of paths through the fixed-decision node s at time t.

We show these figures in Table 5. Moreover, due to lack of space, only four cases, case 1, case 3, case 5, case 7 are shown for the optimal investment units(Table 4) and the average investment proportions(Table 5).

Table 5: Average investment proportions

case 1

cash			stock			bond			CB		
0	1	2	0	1	2	0	1	2	0	1	2
80.76%	77.23%	64.95%	1.17%	0.00%	0.00%	11.10%	14.17%	35.04%	6.97%	8.59%	0.00%
		62.75%			0.77%			32.67%			3.81%
		84.30%			0.00%			0.00%			15.70%
	79.21%	66.03%		0.00%	0.00%		0.00%	0.00%		20.79%	33.97%
		88.83%			11.16%			0.00%			0.00%
		94.27%			5.73%			0.00%			0.00%
	78.74%	93.05%		4.33%	0.60%		16.93%	6.34%		0.00%	0.00%
		64.55%			1.00%			0.01%			34.44%
		77.65%			0.00%			19.04%			3.31%

case 3

cash			stock			bond			CB		
0	1	2	0	1	2	0	1	2	0	1	2
56.57%	91.38%	4.58%	0.06%	0.01%	1.90%	31.68%	2.11%	89.29%	11.69%	6.51%	4.23%
		56.09%			0.38%			41.37%			2.15%
		74.79%			0.00%			0.01%			25.19%
	62.02%	60.07%		0.38%	0.23%		0.00%	0.02%		37.60%	39.69%
		84.76%			15.12%			0.06%			0.05%
		98.07%			1.92%			0.01%			0.00%
	39.41%	46.85%		6.00%	0.00%		54.59%	53.15%		0.00%	0.00%
		7.08%			29.85%			0.00%			63.07%
		95.51%			0.02%			4.42%			0.06%

case 5

cash			stock			bond			CB		
0	1	2	0	1	2	0	1	2	0	1	2
22.61%	85.22%	3.00%	3.12%	0.00%	0.00%	54.64%	0.15%	91.28%	19.63%	14.63%	5.71%
		36.53%			3.55%			59.90%			0.01%
		64.03%			0.00%			0.00%			35.97%
	18.79%	38.23%		0.01%	0.01%		0.00%	0.00%		81.20%	61.76%
		77.20%			22.79%			0.01%			0.01%
		86.88%			13.11%			0.00%			0.00%
	1.75%	2.05%		3.20%	0.00%		95.05%	97.94%		0.00%	0.00%
		7.49%			28.81%			0.00%			63.70%
		78.00%			0.00%			22.00%			0.01%

case 7

cash			stock			bond			CB		
0	1	2	0	1	2	0	1	2	0	1	2
0.00%	44.28%	5.24%	4.73%	0.00%	0.00%	57.35%	16.91%	92.61%	37.92%	38.81%	2.15%
		2.20%			4.24%			73.70%			19.87%
		4.49%			0.00%			0.00%			95.51%
	3.15%	3.08%		10.59%	0.22%		0.00%	0.00%		86.26%	96.70%
		40.72%			59.27%			0.00%			0.00%
		67.99%			32.01%			0.00%			0.00%
	2.88%	3.01%		1.11%	0.00%		96.01%	96.99%		0.00%	0.00%
		8.94%			91.06%			0.00%			0.00%
		2.16%			0.00%			97.78%			0.06%

(2) Expected wealth and LPM_1 values

We show the expected wealth and LPM_1 values in Table 6, and the cumulative discrete distribution of the terminal wealth in Figure 4.

Table 6: Expected wealth and LPM_1 values

	initial wealth	intermediate wealth		terminal wealth	LPM_1
	0	1	2	3	
case 1	10000.0	10049.5	10102.3	10161.1	0.00
case 2	10000.0	10052.2	10106.4	10180.0	1.99
case 3	10000.0	10055.3	10115.5	10195.0	5.44
case 4	10000.0	10059.2	10126.5	10210.0	10.89
case 5	10000.0	10064.4	10139.4	10225.0	18.23
case 6	10000.0	10069.7	10148.8	10240.0	28.26
case 7	10000.0	10072.6	10155.6	10255.0	43.38
case 8	10000.0	10082.4	10174.8	10280.4	127.25

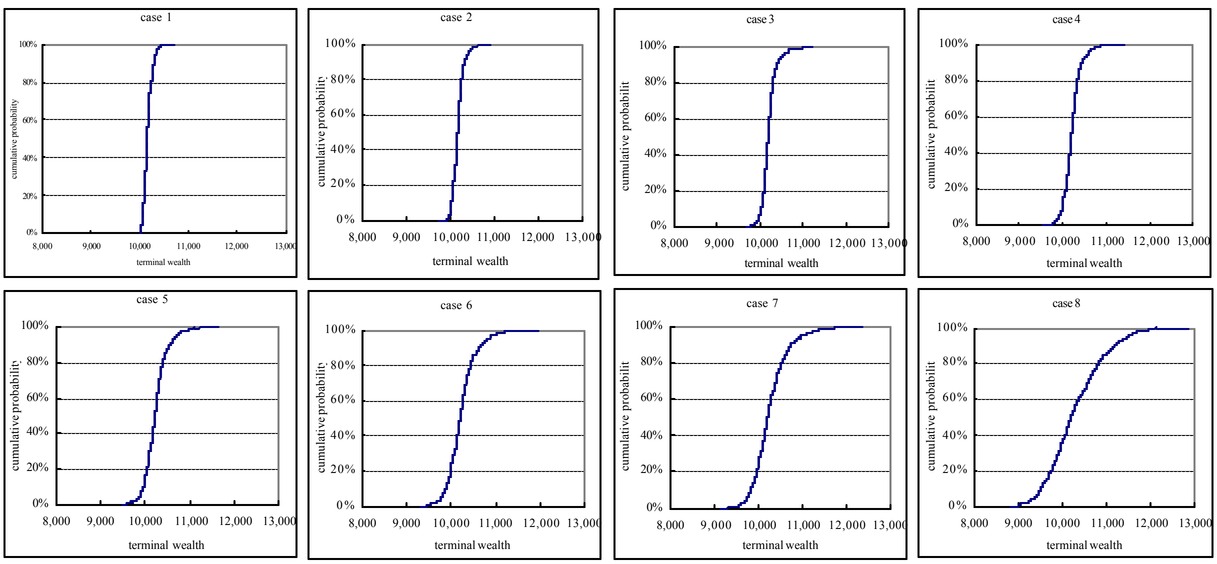

Figure 4: Cumulative discrete distribution of the terminal wealth

In case 1, because risk is minimized, investors should have more cash. As the required expected terminal wealth increases, investors should decrease cash, and more money to risky assets. However, stock which is most risky among three risky assets is not always invested in all fixed-decision nodes. The reasons are:

(1) Investors can select investment decisions according to asset price changes because of the conditional decisions.

(2) Due to the limitation of the model description, the degree of uncertainties decreases as working forward from the initial decision node.

We can find the trade-off between risk and return in Table 6. We find that the larger the required expected terminal wealth is, the larger the volatility is. We can control the risk and return of the terminal wealth in some degree directly by using this model.

4.2 Example 2 : Different branching number of decision nodes

(1) Optimal solutions

We show the optimal results for four kinds of branching, such as 2,3,4 and 5 branching in the case 5. We also show the result for no branching(1 branching), or the simulated path model.

Table 7: Optimal investment units

1 branching

cash			stock			bond			CB		
0	1	2	0	1	2	0	1	2	0	1	2
0.0	85.8	124.6	1864.9	2273.7	1544.9	2909.3	3760.3	3773.5	5225.8	3882.0	4560.9

2 branching

cash			stock			bond			CB		
0	1	2	0	1	2	0	1	2	0	1	2
0.5	5763.7	539.1	353.8	0.2	811.1	7501.6	0.1	8658.2	2144.2	4261.4	0.2
		5638.7			0.1			0.2			4410.1
	195.5	182.2		3.1	0.1		9711.3	9727.8		0.1	0.1
		748.8			0.8			1392.5			7926.6

3 branching

cash			stock			bond			CB		
0	1	2	0	1	2	0	1	2	0	1	2
2261.0	8579.1	302.6	312.4	0.0	0.2	5463.7	14.6	9105.6	1962.9	1460.5	581.5
		3702.6			349.2			6007.5			0.8
		6510.9			0.0			0.1			3484.5
	1916.5	4067.3		0.5	0.7		0.0	0.2		7863.5	5946.9
		7996.3			2149.0			0.7			0.5
		8715.8			1263.9			0.1			0.0
	174.2	204.6		338.2	0.0		9430.5	9707.5		0.0	0.0
		759.8			2878.7			0.0			6319.4
		7838.0			0.1			2182.0			0.5

4 branching

cash			stock			bond			CB		
0	1	2	0	1	2	0	1	2	0	1	2
3468.9	7473.8	402.3	3.5	5.2	4611.3	4730.4	20.4	5088.0	1797.3	2543.5	1.0
		190.0			0.4			8262.9			1488.5
		2761.9			0.1			7281.4			1.0
		430.7			1.0			0.9			9308.5
	9162.2	5974.5		0.2	0.7		105.6	3542.5		800.1	562.0
		7347.7			0.4			0.9			2639.2
		239.8			603.0			9228.8			1.1
		347.2			0.5			9653.3			2.5
	4043.6	5305.7		0.5	11.5		0.2	1.1		5809.2	4687.5
		8246.2			1857.8			3.5			2.6
		2534.3			2326.8			5095.5			0.8
		10080.7			0.6			0.5			0.3
	254.1	159.0		325.7	0.1		9389.0	9788.4		0.1	0.1
		187.6			593.2			9234.5			0.6
		779.8			3210.9			0.2			5989.8
		9360.8			0.5			705.7			0.5

5 branching

cash			stock			bond			CB		
0	1	2	0	1	2	0	1	2	0	1	2
4149.0	7756.6	233.5	1.3	430.9	1777.5	4465.0	11.9	8020.4	1384.7	1848.7	1.0
		391.8			8158.2			0.6			1492.6
		205.3			0.8			8727.2			977.0
		3508.4			0.2			6546.0			2.0
		471.7			1.6			1.5			9269.4
	9603.5	10130.7		0.2	0.7		44.7	4.0		420.4	3.0
		3106.5			5.6			6250.9			671.9
		7708.9			0.6			1.2			2277.5
		250.2			380.2			9438.7			1.5
		349.0			0.8			9653.5			3.1
	5504.1	10376.5		1.5	1.9		0.3	2.1		4400.3	2.7
		1218.3			5185.7			1.7			2725.6
		8463.4			1618.3			5.1			3.7
		1760.1			2195.4			6021.0			1.0
		10086.2			2.2			0.9			0.5
	699.4	2093.2		274.1	0.1		9006.7	7883.1		0.2	0.2
		166.3			1045.7			8886.3			0.7
		507.3			2269.3			0.2			7218.0
		7041.7			2925.0			7.9			1.8
		6149.3			2.3			3895.8			0.3
	3048.6	9969.1		0.7	0.4		6894.3	2.5		0.6	0.9
		217.6			4.7			2.8			10060.7
		453.3			17.1			6.3			10420.4
		9638.8			1.5			340.2			2.2
		6280.4			4145.9			4.2			2.3

Table 8: Average investment proportions

1 branching

cash			stock			bond			CB		
0	1	2	0	1	2	0	1	2	0	1	2
0.00%	0.85%	1.23%	18.65%	22.76%	15.48%	29.09%	37.56%	37.64%	52.26%	38.83%	45.65%

2 branching

cash			stock			bond			CB		
0	1	2	0	1	2	0	1	2	0	1	2
0.00%	56.99%	5.34%	3.54%	0.00%	8.07%	75.02%	0.00%	86.59%	21.44%	43.01%	0.00%
		54.66%			0.00%			0.00%			45.33%
	1.97%	1.83%		0.03%	0.00%		98.00%	98.17%		0.00%	0.00%
		7.43%			0.01%			14.05%			78.50%

3 branching

cash			stock			bond			CB		
0	1	2	0	1	2	0	1	2	0	1	2
22.61%	85.22%	3.00%	3.12%	0.00%	0.00%	54.64%	0.15%	91.28%	19.63%	14.63%	5.71%
		36.53%			3.55%			59.90%			0.01%
		64.03%			0.00%			0.00%			35.97%
	18.79%	38.23%		0.01%	0.01%		0.00%	0.00%		81.20%	61.76%
		77.20%			22.79%			0.01%			0.01%
		86.88%			13.11%			0.00%			0.00%
	1.75%	2.05%		3.20%	0.00%		95.05%	97.94%		0.00%	0.00%
		7.49%			28.81%			0.00%			63.70%
		78.00%			0.00%			22.00%			0.01%

4 branching

cash			stock			bond			CB		
0	1	2	0	1	2	0	1	2	0	1	2
34.69%	74.55%	4.01%	0.04%	0.05%	44.97%	47.30%	0.20%	51.01%	17.97%	25.20%	0.01%
		1.91%			0.00%			83.79%			14.30%
		27.23%			0.00%			72.76%			0.01%
		4.21%			0.01%			0.01%			95.77%
	90.87%	58.86%		0.00%	0.01%		1.06%	35.38%		8.07%	5.75%
		72.27%			0.00%			0.01%			27.72%
		2.38%			5.55%			92.06%			0.01%
		3.44%			0.00%			96.53%			0.02%
	39.79%	50.54%		0.01%	0.13%		0.00%	0.01%		60.20%	49.32%
		80.11%			19.82%			0.03%			0.03%
		25.38%			23.13%			51.48%			0.01%
		99.99%			0.01%			0.01%			0.00%
	2.55%	1.59%		3.07%	0.00%		94.38%	98.41%		0.00%	0.00%
		1.88%			5.12%			93.00%			0.01%
		7.67%			32.07%			0.00%			60.25%
		92.90%			0.00%			7.09%			0.01%

5 branching

cash			stock			bond			CB		
0	1	2	0	1	2	0	1	2	0	1	2
41.49%	77.33%	2.33%	0.01%	4.24%	17.05%	44.65%	0.12%	80.61%	13.85%	18.31%	0.01%
		3.89%			81.41%			0.01%			14.70%
		2.06%			0.01%			88.54%			9.39%
		34.59%			0.00%			65.39%			0.02%
		4.61%			0.02%			0.02%			95.36%
	95.31%	99.92%		0.00%	0.01%		0.45%	0.04%		4.24%	0.03%
		30.65%			0.06%			62.44%			6.86%
		76.00%			0.01%			0.01%			23.98%
		2.48%			3.50%			94.01%			0.01%
		3.46%			0.01%			96.51%			0.03%
	54.28%	99.93%		0.02%	0.02%		0.00%	0.02%		45.70%	0.03%
		11.65%			59.31%			0.02%			29.02%
		82.57%			17.34%			0.05%			0.04%
		17.58%			21.76%			60.65%			0.01%
		99.96%			0.02%			0.01%			0.01%
	6.99%	20.91%		2.60%	0.00%		90.41%	79.09%		0.00%	0.00%
		1.66%			9.08%			89.25%			0.01%
		4.99%			22.49%			0.00%			72.52%
		69.04%			30.86%			0.08%			0.02%
		60.84%			0.02%			39.13%			0.00%
	30.78%	99.96%		0.01%	0.00%		69.21%	0.03%		0.01%	0.01%
		2.18%			0.04%			0.03%			97.75%
		4.53%			0.14%			0.06%			95.26%
		96.55%			0.01%			3.42%			0.02%
		63.56%			36.38%			0.04%			0.02%

(2) Expected wealth and LPM_1 values

We show the expected wealth and LPM_1 values in Table 9, and the cumulative discrete distribution of the terminal wealth in Figure 5.

Table 9: Expected wealth and LPM_1 values

branch	initial wealth	intermediate wealth		terminal wealth	LPM_1
	0	1	2	3	
1	10000.0	10075.1	10150.1	10225.0	97.0
2	10000.0	10069.6	10141.5	10225.0	31.7
3	10000.0	10064.4	10139.4	10225.0	18.2
4	10000.0	10061.0	10132.0	10225.0	14.7
5	10000.0	10058.9	10125.6	10225.0	10.5

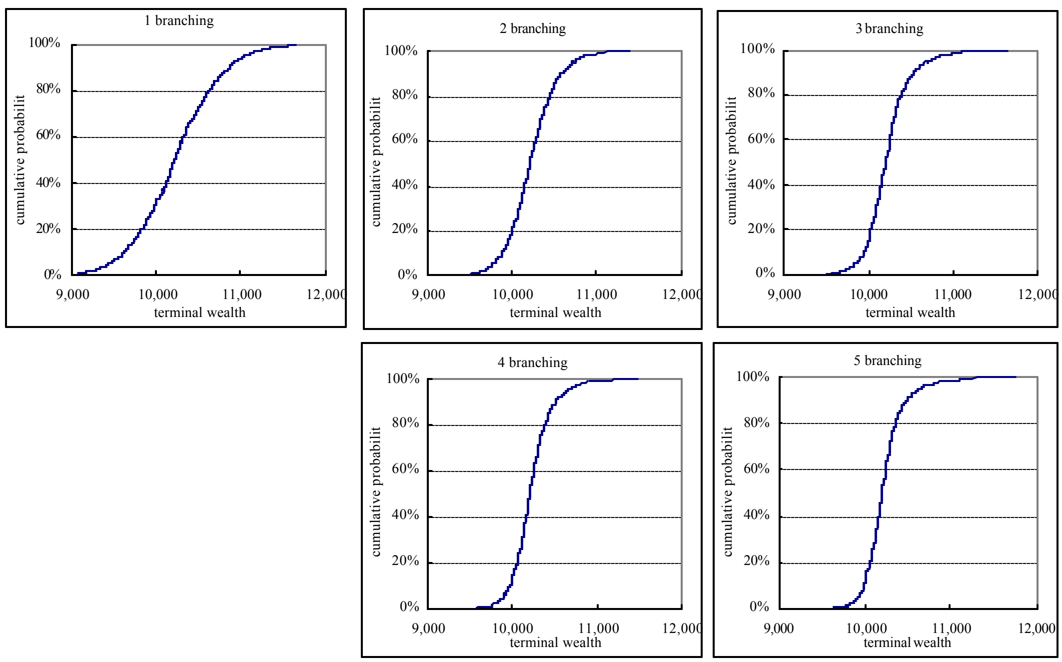

Figure 5: Cumulative discrete distribution of the terminal wealth

As the number of branching increases, investors can have more cash and bond, which are less risky than stock and CB at initial time. The reason is that investors can have more opportunities of the conditional decisions, and therefore can select whether they invest more risky assets or not, according to asset price changes and wealth level as working forward from initial time. As the result, investors can have lower risk under the same expected wealth.

Efficient frontiers in Figure 6 show these results explicitly. As the number of branching increases, we can have more opportunities of the conditional decisions, and we have lower risk under the same expected wealth. If we does not increase the number of simulated paths , note that we may have the low degree of uncertainties due to model description.

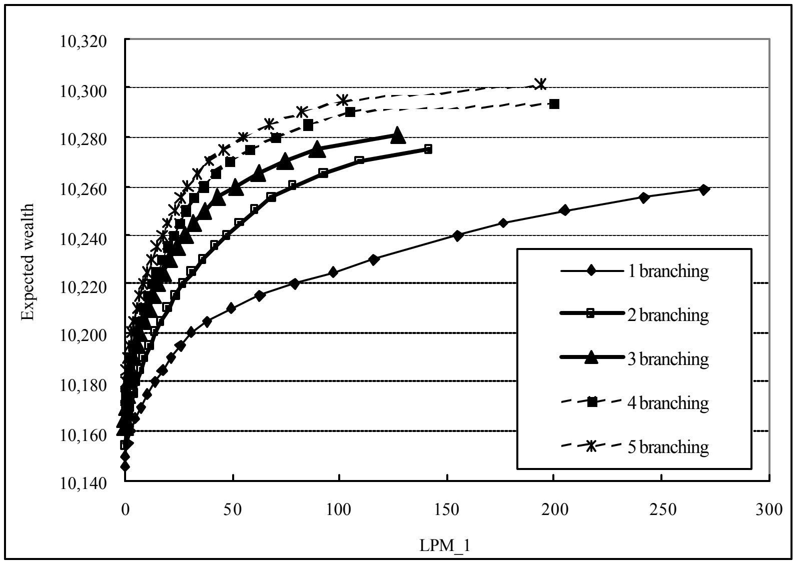

Figure 6: Efficient frontiers

5 Compact representation

We discuss the equivalent formulation to decrease the problem size and its computation time. According to the modeling structure, the number of the decision variables of cash must depends on the set of paths and periods, and then leads to large-scale problems. On the other hand, the number of decision variables of any risky asset depends on only the number of decision nodes. The original formulation is easy to understand the hybrid model, but it does not make the effective use of the decision rule with fixed-unit strategy for risky assets at each nodes. Then we propose an equivalent and a more compact formulation than the original formulation by eliminating cash variables. This does not mean that cash is excluded from the asset allocation decision. We call this formulation "compact representation", and show two kinds of formulations; primal compact representation and dual compact representation. We expect that computation time decreases by reducing the problem size.

5.1 Primal compact representation

(1) Elimination of cash variable v_0 at initial time

We solve Equation (11) for v_0, and it is described by:

$$v_0 = W_0 - \sum_{j=1}^{n} \rho_{j0} z_{j0} \tag{21}$$

We can derive the following constraints by using the non-negative condition of v_0 (Equation (19)).

$$\sum_{j=1}^{n} \rho_{j0} z_{j0} \leq W_0 \tag{22}$$

(2) Elimination of cash variables $v_t^{(i)}$ for path i and time t

We substitute the right-hand side of Equation (21) for v_0 of Equation (12), and then the following equation is obtained.

$$v_1^{(i)} = -\sum_{j=1}^{n} \rho_{j1}^{(i)} h^{(i)}(z_{j1}^{s_1}) - \sum_{j=1}^{n} \left\{ (1+r_0)\rho_{j0} - \rho_{j1}^{(i)} \right\} z_{j0} + (1+r_0)W_0, \quad (i = 1, \ldots, I) \quad (23)$$

Likewise, we rewrite the equation for $v_2^{(i)}$ as follows:

$$\begin{aligned} v_2^{(i)} &= -\sum_{j=1}^{n} \rho_{j2}^{(i)} h^{(i)}(z_{j2}^{s_2}) - \sum_{j=1}^{n} \left\{ \left(1+r_1^{(i)}\right) \rho_{j1}^{(i)} - \rho_{j2}^{(i)} \right\} h^{(i)}(z_{j1}^{s_1}) \\ &\quad - \sum_{j=1}^{n} \left(1+r_1^{(i)}\right) \left\{ (1+r_0)\rho_{j0} - \rho_{j1}^{(i)} \right\} z_{j0} + \left(1+r_1^{(i)}\right)(1+r_0)W_0, \\ &\qquad (s_2 \in S_2;\ i \in V_2^{s_2}) \end{aligned} \quad (24)$$

In general, $v_t^{(i)}$ can be described by:

$$v_t^{(i)} = -\sum_{j=1}^{n} \rho_{jt}^{(i)} h^{(i)}(z_{jt}^{s_t}) + \sum_{j=1}^{n} \sum_{k=0}^{t-1} \eta_{jkt}^{(i)} h^{(i)}(z_{jk}^{s_k}) + F_t^{(i)}, \ (t = 1, \ldots, T-1;\ s_t \in S_t;\ i \in V_t^{s_t}) \quad (25)$$

where

$$\begin{aligned} \eta_{j,k,k+1}^{(i)} &= \rho_{j,k+1}^{(i)} - \left(1+r_k^{(i)}\right) \rho_{jk}^{(i)}, \quad (k = 0, \ldots, T-1) \\ \eta_{jkt}^{(i)} &= \left(1+r_{t-1}^{(i)}\right) \eta_{j,k,t-1}^{(i)}, \quad (t = k+2, \ldots, T) \\ F_1^{(i)} &= (1+r_0)\, W_0 \\ F_t^{(i)} &= \left(1+r_{t-1}^{(i)}\right) F_{t-1}^{(i)}, \quad (t = 2, \ldots, T). \end{aligned}$$

Therefore, we rewrite Equation (20) as follows:

$$\sum_{j=1}^{n} \rho_{jt}^{(i)} h^{(i)}(z_{jt}^{s_t}) - \sum_{j=1}^{n} \sum_{k=0}^{t-1} \eta_{jkt}^{(i)} h^{(i)}(z_{jk}^{s_k}) \le F_t^{(i)}, \quad (t = 1, \ldots, T-1;\ s_t \in S_t;\ i \in V_t^{s_t}) \quad (26)$$

We explain the meanings of $\eta_{j,k,k+1}^{(i)}$, $\eta_{jkt}^{(i)}$, and $F_t^{(i)}$. $\eta_{j,k,k+1}^{(i)}$ is rearranged as follows;

$$\eta_{j,k,k+1}^{(i)} = \left(1+\mu_{j,k+1}^{(i)}\right) \rho_{jk}^{(i)} - \left(1+r_k^{(i)}\right) \rho_{jk}^{(i)} = \left(\mu_{j,k+1}^{(i)} - r_k^{(i)}\right) \rho_{jk}^{(i)} \quad (27)$$

Equation (27) shows that $\eta_{j,k,k+1}^{(i)}$ is the excess rate of return when asset j is invested at time k in one period, where $\mu_{j,k+1}^{(i)}$ is the rate of return of risky asset j in period $k+1$(from time k to $k+1$). $\eta_{jkt}^{(i)}$ is the cumulative rate of return when this excess return $\eta_{j,k,k+1}^{(i)}$ is invested by cash from time $k+1$ to t. For example, $\eta_{j01}^{(i)} z_{j0}$ is the excess return when z_{j0} is invested in period 1, and after this is invested from time 1 to t by cash, we can obtain the cumulative return $\eta_{j0t}^{(i)} z_{j0}$ at time t.

$F_t^{(i)}$ is wealth at time t when the initial wealth W_0 is invested by cash for t periods.

(3) Wealth $W_t^{(i)}$ for path i and time t

Both sides of Equation (13) show the wealth $W_t^{(i)}$ for path i and time t. We calculate

$$W_t^{(i)} = \sum_{j=1}^{n} \sum_{k=0}^{t-1} \eta_{jkt}^{(i)} h^{(i)}(z_{jk}^{s_k}) + F_t^{(i)}, \ (t = 1, \ldots, T-1;\ s_t \in S_t;\ i \in V_t^{s_t}) \quad (28)$$

by using the right-hand side of Equation (13). Especially, we write

$$W_T^{(i)} = \sum_{j=1}^{n} \sum_{k=0}^{T-1} \eta_{jkT}^{(i)} h^{(i)}(z_{jk}^{s_k}) + F_T^{(i)}, \; (s_T \in S_T; \; i \in V_T^{s_T}) \tag{29}$$

for $t = T$. We find that wealth at time t is divided into two parts; the first part shows the summation of one period excess returns which are invested by cash until time t, and the second part shows the cumulative return in period t in cash investment.

(4) Formulation

We write the formulation by primal compact representation as follows:

$$\textbf{Minimize} \quad \frac{1}{I} \sum_{i=1}^{I} q^{(i)} \tag{30}$$

subject to

$$\sum_{j=1}^{n} \rho_{j0} z_{j0} \leq W_0 \tag{31}$$

$$\sum_{j=1}^{n} \rho_{jt}^{(i_t)} h^{(i)}(z_{jt}^{s_t}) - \sum_{j=1}^{n} \sum_{k=0}^{t-1} \eta_{jkt}^{(i_k)} h^{(i)}(z_{jk}^{s_k}) \leq F_t^{(i_t)}, \; (t = 1, \ldots, T-1; \; s_t \in S_t; \; i_t \in V_t^{s_t}) \tag{32}$$

$$\sum_{j=1}^{n} \sum_{k=0}^{T-1} \sum_{s_k \in S_k} \eta_{jkT}^{(i_k)} h^{(i)}(z_{jk}^{s_k}) + q^{(i_{T-1})} \geq W_G - F_T^{(i_{T-1})}, \; (s_{T-1} \in S_{T-1}; \; i_{T-1} \in V_{T-1}^{s_{T-1}}) \tag{33}$$

$$\frac{1}{I} \sum_{j=1}^{n} \sum_{k=0}^{T-1} \sum_{s_k \in S_k} \sum_{i_k \in V_k^{s_k}} \eta_{jkT}^{(i_k)} h^{(i_k)}(z_{jk}^{s_k}) \geq W_E - \overline{F}_T \tag{34}$$

$$z_{j0} \geq 0, \quad (j = 1, \ldots, n)$$

$$z_{jt}^{s_t} \geq 0, \quad (j = 1, \ldots, n; \; t = 1, \ldots, T-1 \; ; s_t \in S_t)$$

$$q^{(i)} \geq 0, \quad (i = 1, \ldots, I)$$

where $\overline{F}_T = \frac{1}{I} \sum_{i=1}^{I} F_T^{(i)}$. i_t denotes the path through the decision node S_t at time t.

Table 10: Primal form for $h^{(i)}(z_{jt}^s) = z_{jt}^s$ and $T = 4$

Equation	z_{j0}	$z_{j1}^{s_1}$	$z_{j2}^{s_2}$	$z_{j3}^{s_3}$	$q^{(i)}$	sign	RHS	dual	No. of Const.
(30)[**Min**]	0	0	0	0	$\frac{1}{I}$				
(31)	$-\rho_{j0}$	0	0	0	0	$\geq$	$-W_0$	λ_0	1
(32), $t=1$	$\eta_{j01}^{(i)}$	$-\rho_{j1}^{(i_1)}$	**o**	**o**	**o**	$\geq$	$-F_1^{(i)}$	$\lambda_1^{(i)}$	I
(32), $t=2$	$\eta_{j02}^{(i)}$	$\eta_{j12}^{(i_1)}$	$-\rho_{j2}^{(i_2)}$	**o**	**o**	$\geq$	$-F_2^{(i)}$	$\lambda_2^{(i)}$	I
(32), $t=3$	$\eta_{j03}^{(i)}$	$\eta_{j13}^{(i_1)}$	$\eta_{j23}^{(i_2)}$	$-\rho_{j3}^{(i_3)}$	**o**	$\geq$	$-F_3^{(i)}$	$\lambda_3^{(i)}$	I
(33)	$\eta_{j0T}^{(i)}$	$\eta_{j1T}^{(i_1)}$	$\eta_{j2T}^{(i_2)}$	$\eta_{j3T}^{(i_3)}$	$\boldsymbol{E}$	$\geq$	$W_G - F_T^{(i)}$	$\lambda_T^{(i)}$	I
(34)	$\overline{\eta}_{j0T}$	$\overline{\eta}_{j1T}^{s_1}$	$\overline{\eta}_{j2T}^{s_2}$	$\overline{\eta}_{j3T}^{s_3}$	0	$\geq$	$W_E - \overline{F}_T$	ω	1

† $\boldsymbol{E}$: unit matrix, $\overline{\eta}_{j0T} = \frac{1}{I} \sum_{i=1}^{I} \eta_{jkT}^{(i)}$

5.2 Dual compact representation

For simple description, we denote investment units by $h^{(i)}(z^s_{jt}) = z^s_{jt}$, and we write the dual form by using the primal form as in Table 10.

The dual compact formulation of the hybrid model is decribed as follows:

$$\textbf{Maximize} \quad -W_0\lambda_0 - \sum_{i=1}^{I}\sum_{t=1}^{T-1} F_t^{(i)}\lambda_t^{(i)} + \sum_{i=1}^{I}\left(W_G - F_T^{(i)}\right)\lambda_T^{(i)} + \left(W_E - \overline{F}_T\right)\omega \tag{35}$$

subject to

$$-\rho_{j0}\lambda_0 + \sum_{i=1}^{I}\sum_{t=1}^{T}\eta_{j0t}^{(i)}\lambda_t^{(i)} + \overline{\eta}_{j0T}\omega \le 0, \; (j = 1, \ldots, n) \tag{36}$$

$$-\sum_{i_k \in V_k^{s_k}} \rho_{jk}^{(i_k)}\lambda_k^{(i_k)} + \sum_{i_k \in V_k^{s_k}} \sum_{t=k+1}^{T} \eta_{jkt}^{(i_k)}\lambda_t^{(i_k)} + \overline{\eta}_{jkT}^{s_k}\omega \le 0,$$

$$(j = 1, \ldots, n; \; k = 1, \ldots, T-1; \; s_k \in S_k) \tag{37}$$

$$\lambda_T^{(i)} \le \frac{1}{I}, \; (i = 1, \ldots, I) \tag{38}$$

$$\lambda_0 \ge 0$$

$$\lambda_t^{(i)} \ge 0, \quad (t = 1, \ldots, T; \; i = 1, \ldots, I)$$

$$\omega \ge 0$$

where $\overline{\eta}_{jkT}^{s_k} = \frac{1}{I}\sum_{i_k \in V_k^{s_k}} \eta_{jkT}^{(i_k)}$, $(k = 1, \ldots, T-1)$. Dual variables are:

Equation (36) : z_{j0} , $(j = 1, \ldots, n)$

Equation (37) : $z_{jk}^{s_k}$, $(j = 1, \ldots, n; \; k = 1, \ldots, T-1; \; s_k \in S_k)$

Equation (38) : $q^{(i)}$, $(i = 1, \ldots, I)$

In the dual compact representation, we have $TI+2$ decision variables, and $nT+I$ constraints. Because the piecewise linear risk measure, such as the first-order lower partial moment, is used, the number of boundary constraints is the number of sample paths, and the number of the general constraints depends on only the number of decision nodes and risky assets. Therefore, we have only nT general constraints. The problem size associated with the computation time is drastically shrunk [10] . We expect that computation time decreases by this structure.

5.3 Comparison of computation time

We examine several numerical examples in order to compare the computation time of the dual compact formulation with that of original formulation. Three periods and four assets problems are solved using NUOPT Ver.4 for four kinds of branching number and six kinds of sample path number. We use Windows 98 personal computer which has 1500 MHz CPU and 512MB memory.

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[10] In general, linear programming problem is formulated as follows;

$$\textbf{Minimize} \quad c^T x \tag{39}$$

$$\textbf{subject to} \quad \underline{b} \le Ax \le \overline{b} \tag{40}$$

$$l \le x \le u \tag{41}$$

We have two kinds of constraints in this problem; general constraints (40) and boundary constraints (41). The bounded variables are specially treated in the algorithm and the mathematical programming software, and therefore the boundary constraints are treated differently from the general constraints.

We show computation time for two kinds of formulations in Figure 7 and in Table 11. The left side figures show the computation time by the interior point method, and the right side figures show the computation time by the simplex method.

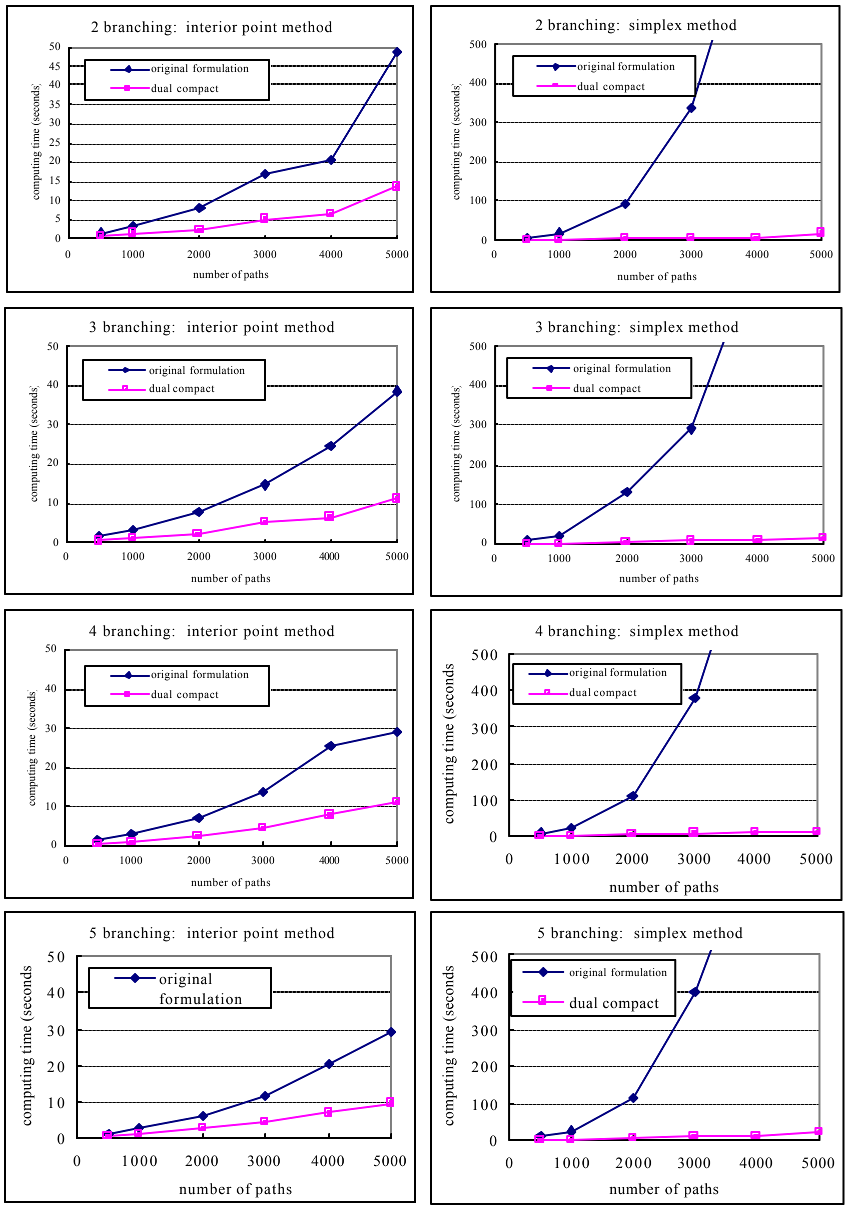

Figure 7: Comparison of the computation time

Table 11: Computation time

Interior point method							(unit: seconds)
	path	500	1,000	2,000	3,000	4,000	5,000
2 branching	original	1.43	3.40	7.96	16.86	20.27	48.88
	dual	0.55	1.43	2.30	5.22	6.42	13.56
3 branching	original	1.31	3.24	7.69	14.50	24.49	38.61
	dual	0.44	1.21	2.31	4.94	6.59	10.99
4 branching	original	1.32	2.80	7.25	14.01	25.49	28.78
	dual	0.49	1.10	2.36	4.67	8.02	11.21
5 branching	original	1.15	2.58	6.26	11.42	20.27	29.17
	dual	0.33	1.05	2.58	4.07	6.81	9.45
Simplex method							(unit: seconds)
	path	500	1,000	2,000	3,000	4,000	5,000
2 branching	original	5.00	18.89	93.81	336.20	885.34	869.36
	dual	0.22	0.60	2.26	5.11	8.73	17.74
3 branching	original	5.99	21.14	131.11	292.48	738.20	1,623.48
	dual	0.33	0.94	1.92	5.33	9.34	15.87
4 branching	original	6.92	22.74	105.95	380.69	891.77	1,360.12
	dual	0.55	1.09	2.52	7.58	12.09	12.74
5 branching	original	5.99	22.35	113.92	403.37	846.85	1,487.93
	dual	0.39	0.88	3.95	7.47	12.42	20.65

Figure 8 shows original/dual ratios; computation time of the original formulation to computation time of the dual compact formulation. When the interior point method is used, the dual compact formulation can be solved about three times as fast as the original formulation as in the left side of Figure 8. When the dual form is solved by using the simplex method, the computation time can be improved drastically as in the right side of Figure 8.

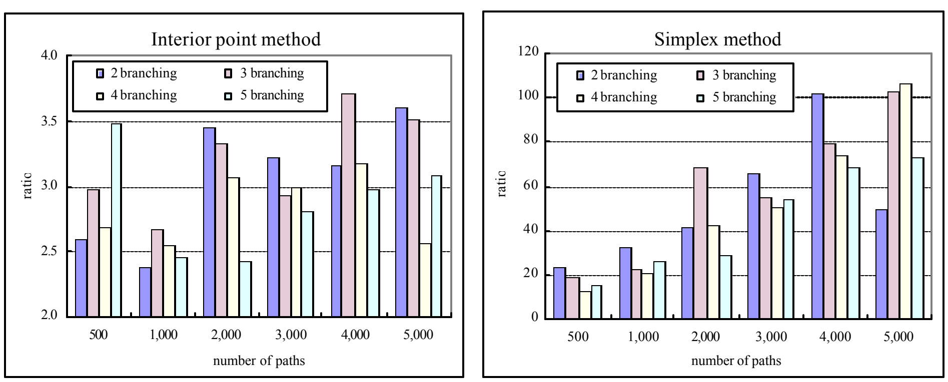

Figure 8: Original/dual ratio

6 Conclusion

We show the hybrid optimization model by using simulated paths and the decision tree in order to do both describing the accuracy of uncertainties and making the conditional decisions. We have sample paths associated with asset returns using Monte Carlo simulation, while we allow the model to expand the decision space and to make the conditional decisions as in the scenario tree model. We show the technique of generating the extended decision tree to make the conditional decisions in the simulated path framework, which is called the sequential clustering method.

We test some cases using numerical examples. Because the hybrid model can be formulated as the linear programming problem, it can be solved very fast for the large scale problem in practice. According to some numerical tests, we find that we can control the risk and return of the terminal wealth in some degree directly.

The hybrid model can be formulated as its compact representation form to reduce the program size. We show the computation time for two kinds of formulations; the original formulation and the dual compact formulation. We find that we can decrease its computation time, according to some numerical tests.

This model is developed for asset allocation, but the idea of this modeling can be widely applied to the financial problem, such as optimal ALM problem, optimal bond portfolio selection, and so on. In our future research, we will model the pension fund ALM, banking ALM in the hybrid modeling framework. Furthermore, we must also investigate the characteristics of this model using additional numerical tests.

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